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Equations polyharmoniques sur les variétés et études asymptotiques dans une équation de Hardy-Sobolev

THÈSE

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Contents

CHAPTER 1

Introduction

Nonlinear problems such as those that naturally arise from geometry and physics like the study of geodesics, minimal surfaces, harmonic maps, conformal metrics with prescribed curvature, Hamiltonian systems, solutions of boundary value problems and Yang-Mills fields, can all be characterised as critical points u of some functional F on an appropriate space X, i.e., $F'(u) = 0$. So one is concerned with problems of existence, location, multiplicity and qualitative properties of critical points in such contexts and how they relate to the (weak) solutions they represent for the corresponding Euler-Lagrange equations.
The points of maxima or minima, if it exists, are the simplest example of critical points for F. In general the functional F maybe unbounded on X or it may not achieve maximum or the minimum value(s). Locating critical levels for a smooth functional F on a space X essentially reduces to capturing the changes in the topology of the sublevel sets $F_a = \{x \in X : F(x) < a\}$ as a varies in $\mathbb{R}$. Under the right conditions on F, classical Morse theory states that a non-trivial topology between F_a and F_b should detect a critical level c between a and b. The next simplest example short of considering minimisation consists of taking two points u_0 and u_1 both lying below level a which are not connected in F_a, but become so if one can climb above that level. This means that the the sublevels F_a and F_b have different topologies for some $b > a$ and this yields a critical point c between levels a and b. This setting is often called the Mountain-Pass Principle since in practice one insures that the two villages are disconnected below level a by showing that they are separated by a mountain range with minimal altitude exceeding a.

The above proposition identifies a potential critical level. The problem of existence of a critical point then reduces to proving that a sequence (x_m) satisfying $\lim\limits_{m\to\infty} F(x_m) = c$ and $\lim\limits_{m\to\infty} \|F'(x_m)\| = 0$ is relatively compact in X. This is usually where the hard analysis is needed. Any function possessing such a property is said to satisfy the Palais-Smale condition at level c, in short $(PS)_c$.

In this memoir we study some variational elliptic partial differential equations without the compactness property as described above. In problems of these kind one encounters a *blow-up* phenomenon caused by scale and conformal invariance, which makes it *non-compact*. However, this lack of compactness is not always the final word and a finer analysis of the behavior of non-compact sequences may provide us with some new conditions that could prevent such an eventuality. As a model case, one can think of the well studied stationary Schrödinger equation

$$\begin{cases} \Delta u + hu = |u|^{\frac{4}{n-2}} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Delta := -\text{div}(\nabla)$ is the Laplacian operator with negative sign convention, Ω is a bounded smooth domain in $\mathbb{R}^n$ or a closed Riemannian manifold of dimension $n \geq 3$ and $h \in C^1(\Omega)$.

Broadly speaking, this memoir is divided into two parts.

Part 1: We analyse the question of existence for some Polyharmonic boundary value problems with critical Sobolev growth on a compact Riemannian manifold.

Part 2: Here we do a blow-up analysis of the nonlinear elliptic Hardy-Sobolev equation with critical growth and vanishing boundary singularity.

We give a quick overview of these topics, providing also an outline of the content of this memoir.

Part 1

Let M be a closed manifold of dimension $n \geq 3$ and let k be a positive integer such that $2k < n$. In recent years, there have been extensive study of the relationship between the conformally covariant operators, that is, operators which satisfy some invariance property under conformal change of metric on M, their associated conformal invariants, and the study of the related partial differential equations. In their celebrated work Graham-Jenne-Mason-Sparling [**27**] provided a systematic construction of a family of conformally covariant operators (GJMS operators for short) based on the ambient metric of Fefferman-Graham [**18**]. More precisely, let $\mathcal{M}$ be the set of Riemannian metrics on M, then for all g in $\mathcal{M}$, there exists a local differential operator $P_g : C^\infty(M) \to C^\infty(M)$ such that $P_g = \Delta_g^k + l.o.t$ where $\Delta_g := -\text{div}_g(\nabla)$, and, given $u > 0$, $u \in C^\infty(M)$ and defining $\hat{g} = u^{\frac{4}{n-2k}} g$, one has

$$P_{\hat{g}}(\varphi) = u^{-\frac{n+2k}{n-2k}} P_g(u\varphi) \text{ for all } \varphi \in C^\infty(M). \tag{1.1}$$

Moreover, P_g is self-adjoint with respect to the L^2-scalar product. A scalar invariant is associated to this operator, namely the Q-curvature, denoted as Q_g. When $k = 1$, P_g is the conformal Laplacian and the $Q-$curvature is the scalar curvature multiplied by a constant. When $k = 2$, P_g is the Paneitz operator introduced in [**40**]. The Q-curvature was introduced by Branson and Ørsted [**9**] and later generalised by Branson[**7**, **8**]. In the specific case $n > 2k$, we have that $Q_g := \frac{2}{n-2k} P_g(1)$. Then, taking $\varphi \equiv 1$ in (1.1), we get that $P_g u = \frac{n-2k}{2} Q_{\hat{g}} u^{\frac{n+2k}{n-2k}}$ on M. Therefore, prescribing the $Q-$curvature in a conformal class amounts to solving a nonlinear elliptic partial differential equation of $2k^{th}$ order. Results for the prescription of the $Q-$curvature problem for the Paneitz operator (namely $k = 2$) are in Djadli-Hebey-Ledoux [**16**] and Esposito-Robert [**17**] for instance. Recently, Gursky-Malchiodi [**28**] proved the existence of a metric with constant $Q-$curvature (still for $k = 2$) provided certain geometric hypotheses on the manifold (M, g) holds. These hypotheses have been simplified by Hang-Yang [**31**].

This leads us to investigate the existence of $u \in C^\infty(M)$, $u > 0$, given $f \in C^\infty(M)$, such that

$$Pu = f u^{2_k^\sharp - 1} \text{ in } M, \tag{1.2}$$

where $2_k^\sharp := \frac{2n}{n-2k}$ and $P : C^\infty(M) \to C^\infty(M)$ is a smooth self-adjoint $2k$-th order partial differential operator defined by

$$Pu = \Delta_g^k u + \sum_{l=0}^{k-1} (-1)^l \nabla^{j_l \ldots j_1} \left(A_{l\, i_1 \ldots i_l, j_1 \ldots j_l} \nabla^{i_1 \ldots i_l} u \right) \tag{1.3}$$

here the indices are raised via the musical isomorphism and for all $l \in \{0, \ldots, k-1\}$, A_l is a smooth symmetric T_{2l}^0-tensor field on M (that is: $A_l(X,Y) = A_l(Y,X)$ for all T_0^l-tensors X, Y on M). When $P := P_g$, then (1.2) is equivalent to saying that $Q_{\hat{g}} = \frac{2}{n-2k} f$ with $\hat{g} = u^{\frac{4}{n-2k}} g$.

Equation (1.2) has a variational structure. Since P is self-adjoint in L^2, we have that for all $u, v \in C^\infty(M)$.

$$\int_M uP(v)\, dv_g = \int_M vP(u)\, dv_g = \int_M \Delta_g^{k/2} u \Delta_g^{k/2} v\, dv_g + \sum_{l=0}^{k-1} \int_M A_l(\nabla^l u, \nabla^l v)\, dv_g$$

where

$$\Delta_g^{l/2} u := \begin{cases} \Delta_g^m u & \text{if } l = 2m \text{ is even} \\ \nabla \Delta_g^m u & \text{if } l = 2m+1 \text{ is odd} \end{cases}$$

and when $l = 2m+1$ is odd, $\Delta_g^{k/2} u\, \Delta_g^{k/2} v = \left(\nabla \Delta_g^m u, \nabla \Delta_g^m v \right)_g$. If P is coercive and $f > 0$, then, up to multiplying by a constant, any non-trivial solution $u \in C^\infty(M)$ to (1.2) is a critical point of the functional

$$u \mapsto J_P(u) := \frac{\int_M uP(u)\, dv_g}{\left(\int_M f|u|^{2_k^\sharp}\ dv_g \right)^{2/2_k^\sharp}} \tag{1.4}$$

The natural space to study J_P is the Sobolev space $H_k^2(M)$; where for $1 \le l \le k$, $H_l^2(M)$ is the completion of $C^\infty(M)$ with respect to the $u \mapsto \sum_{\alpha=0}^{l} \|\nabla^\alpha u\|_2$. Equivalently (see Robert [**43**]), $H_l^2(M)$ can also be seen as the completion of the space $C^\infty(M)$ with respect to the norm

$$\|u\|_{H_l^2}^2 := \sum_{\alpha=0}^{l} \int_M (\Delta_g^{\alpha/2} u)^2\ dv_g$$

By the Sobolev embedding theorem we get a continuous but not compact embedding of $H_k^2(M)$ into $L^{2_k^\sharp}(M)$. The continuity of the embedding $H_k^2(M) \hookrightarrow L^{2_k^\sharp}(M)$ yields a pair of real numbers A, B such that for all $u \in H_k^2(M)$

$$\|u\|_{L^{2_k^\sharp}}^2 \le A \int_M (\Delta_g^{k/2} u)^2\, dv_g + B\, \|u\|_{H_{k-1}^2}^2 \tag{1.5}$$

Following the terminology introduced by Hebey [**32**], we then define

$$\mathcal{A}(M) := \inf\{A \in \mathbb{R} : \exists\ B \in \mathbb{R} \text{ with the property that inequality (1.5) holds}\}$$

As in the classical case $k = 1$ by Aubin, the value of $\mathcal{A}(M)$ depends only on k and the dimension n. More precisely, we let $\mathscr{D}^{k,2}(\mathbb{R}^n)$ be the completion of $C_c^\infty(\mathbb{R}^n)$ for the norm $u \mapsto \|\Delta^{k/2}u\|_{L^2(\mathbb{R}^n)}$, and define $K_0(n,k) > 0$

$$\frac{1}{K_0(n,k)} := \inf_{u\in\mathscr{D}^{k,2}(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n}(\Delta^{k/2}u)^2\,dx}{\left(\int_{\mathbb{R}^n}|u|^{2^\sharp_k}\,dx\right)^{\frac{2}{2^\sharp_k}}} \tag{1.6}$$

as the best constant in the Sobolev's continuous embedding $\mathscr{D}^{k,2}(\mathbb{R}^n) \hookrightarrow L^{2^\sharp_k}(\mathbb{R}^n)$. It follows from Lions [**36**], Ge-Wei-Zhou [**20**], that the extremal functions for the Sobolev inequality (1.6) exist and are exactly multiples of the functions

$$U_{a,\lambda} = \alpha_{n,k}\left(\frac{\lambda}{1+\lambda^2|x-a|^2}\right)^{\frac{n-2k}{2}} \quad a \in \mathbb{R}^n, \lambda > 0$$

where $\alpha_{n,k}$'s are explicit.

For polyharmonic operators on a compact Riemannian manifold we obtain the following *best constant* result:

Theorem 1.1. (*Mazumdar* [**37**]*, see Chapter* 2) *Let (M,g) be a smooth, compact Riemannian manifold of dimension n and let k be a positive integer such that $2k < n$. Then $\mathcal{A}(M) = K_0(n,k) > 0$. In particular, for any $\epsilon > 0$, there exists $B_\epsilon \in \mathbb{R}$ such that for all $u \in H^2_k(M)$ one has*

$$\left(\int_M |u|^{2^\sharp_k}\,dv_g\right)^{\frac{2}{2^\sharp_k}} \le (K_0(n,k)+\epsilon)\int_M (\Delta_g^{k/2}u)^2\,dv_g + B_\epsilon\,\|u\|^2_{H^2_{k-1}}$$

As a consequence of this result, we obtain a description of noncompact bounded families in $H^2_k(M)$. This is the extension of the PL Lions concentration compactness lemma for Riemannian manifolds:

Theorem 1.2. (*Mazumdar* [**38**]*, see Chapter* 2) *Let (M,g) be a smooth, compact Riemannian manifold of dimension n and let k be a positve integer such that $2k < n$. Suppose (u_m) be a bounded sequence in $H^2_k(M)$ such that $\mu_m \rightharpoonup \mu$ weakly in $\mathcal{M}$*

(a) $u_m \rightharpoonup u$ *weakly in* $H^2_k(M)$

(b) $\mu_m = |\Delta_g^{k/2}u_m|_g^2\,dv_g \rightharpoonup \mu$ *weakly in the sense of measures*

(c) $\nu_m = |u_m|^{2^\sharp_k}\,dv_g \rightharpoonup \nu$ *weakly in the sense of measures*

Then we have:

(i) *There exists an at most countable index set $\mathcal{I}$, a family of distinct points $\{x_i \in M : i \in \mathcal{I}\}$, families of nonnegative weights $\{\alpha_i : i \in \mathcal{I}\}$ and $\{\beta_i : i \in \mathcal{I}\}$ such that*

$$\nu = |u|^{2^\sharp_k} + \sum_{i\in\mathcal{I}} \alpha_i\delta_{x_i} \tag{1.7}$$

$$\mu \ge |\Delta_g^{k/2}u|_g^2 + \sum_{i\in\mathcal{I}} \beta_i\delta_{x_i} \tag{1.8}$$

where δ_x denotes the Dirac measure concentrated at $x \in M$ with mass equal to 1 .

(ii) *In addition we have for all $i \in \mathcal{I}$*

$$\alpha_i^{2/2_k^\sharp} \leq K_0(n,k)\ \beta_i \tag{1.9}$$

In particular $\sum_{i\in\mathcal{I}} \alpha_i^{2/2_k^\sharp} < \infty$.

(iii) *Furthermore, if $u \equiv 0$ and $\nu(M)^{2/2_k^\sharp} \geq K_0(n,k)\ \mu(M)$, then ν is concentrated at a single point.*

Another consequence of $\mathcal{A}(M) = K_0(n,k)$ is the existence of minimum energy solutions to (1.2) when the functional J_P goes below a quantified threshold (see Theorem 1.3 below). In general the conformal covariance of the geometric operator P_g yields obstruction to the existence of solutions to (1.2). In particular, it follows from [**12**] that on the canonical sphere $(\mathbb{S}^n, \mathrm{can})$, there is no positive solution $u \in C^\infty(\mathbb{S}^n)$ to the equation $P_{\mathrm{can}} u = (1+\epsilon\varphi)u^{2_k^\sharp - 1}$, for all $\epsilon \neq 0$ and all first spherical harmonic φ.

We remark that any weak solution to equation (1.2) is infact a classical solution. The proof (Mazumdar [**37**], see Chapter 2) is based on the ideas developed by Van der Vorst [**49**]. Concerning the existence of weak solutions to equation (1.2) we first look for minimizers of the functional J_P. The result we obtain in this direction (in the spirit of Aubin) is

Theorem 1.3. (*Mazumdar* [**37**]*, see Chapter* 2) *Let (M,g) be a compact Riemannian manifold of dimension $n > 2k$, with $k \geq 1$. Let P be a differential operator as in* (1.3) *and let $f \in C^{0,\theta}(M)$ be a Hölder continuous positive function. Assume that P is coercive on $H_{k,0}^2(M)$. Suppose that*

$$\inf_{u\in\mathcal{N}_f} \int_M uP(u)\, dv_g < \frac{1}{(\sup_M f)^{\frac{2}{2_k^\sharp}}\ K_0(n,k)},$$

where

$$\mathcal{N}_f := \{u \in H_k^2(M) : \int\limits_M f\,|u|^{2_k^\sharp}\, dv_g = 1\}$$

Then there exists a minimizer $u \in \mathcal{N}_f$. Moreover, up to multiplication by a constant $u \in C^{2k}(M)$ is a solution to

$$Pu = fu^{2_k^\sharp - 1} \text{ in } M.$$

In addition, if the Green's function of P on M with Dirichlet boundary condition is positive, then upto changing sign $u > 0$ is a classical solution to

$$Pu = fu^{2_k^\sharp - 1} \text{ in } M.$$

In a remarkable result first Coron [**11**], and then Bahri-Coron [**5**] showed that the topology of the domain plays a role in proving the existence of solutions to equations like (1.11) when $k = 1$. Coron [**11**] showed that the equation

$$\begin{cases} \Delta u = u^{\frac{n+2}{n-2}} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

always admits a solution if the bounded smooth domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, has a sufficiently small hole. Here $\Delta = -\sum_i \partial_{ii}$ is the Laplacian with minus sign convention. The idea of the proof is to argue by contradiction and to use a minimax method for the corresponding energy functional J, based on a set T of non-negative functions which are homeomorphic to the $(n-1)$ dimensional sphere Σ around a point in Ω. The set T is contractible in the positive cone in $H^2_{1,0}(\Omega)$. So if the above equation does not admit a solution, then under certain conditions such a contraction of T in $H^2_{1,0}(\Omega)$ will induce a contraction of Σ in Ω, giving the desired contradiction.

Infact Bahri-Coron [**5**] showed that the effect of topology is much stronger, and extended the Coron's result for the case when Ω has a non-trivial topology(homology). We note that the solutions obtained by these topological methods are in general not a minimiser of the corresponding energy functional J_P. The result of Coron [**11**] has been generalised for the polyharmonic case by Ge and al. [**20**], and Weth and al. [**6**] for domains in $\mathbb{R}^n$. The next theorem proved in [**37**] is in this spirit:

Theorem 1.4. (*Mazumdar* [**37**]*, see Chapter* 2) *Let (M,g) be a smooth, compact Riemannian manifold of dimension n and let k be a positive integer such that $2k < n$. We let P be a coercive operator as in* (1.3)*. Let $\iota_g > 0$ be the injectivity radius of the manifold M. Suppose that the manifold M contains a point x_0 such that the embedded $(n-1)$ dimensional sphere $\mathbb{S}_{x_0}(\iota_g/2) := \{x \in M /\, d_g(x,x_0) = \iota_g/2\}$ is not contractible in $M \backslash \{x_0\}$. Then there exists $\epsilon_0 \in (0, \frac{\iota_g}{2})$ such that the equation*

$$\begin{cases} Pu = |u|^{2^\sharp_k - 2} u & \text{in } \Omega_M \\ D^\alpha u = 0 & \text{on } \partial\Omega_M \quad \text{for } |\alpha| \leq k-1 \end{cases} \tag{1.10}$$

has a non-trivial $C^{2k}(\Omega_M)$ solution for $\Omega_M := M \backslash \overline{B}_{x_0}(\epsilon_0)$. Moreover, if the Green's Kernel of P on Ω_M is positive, then we can choose $u > 0$.

In the original result of Coron [**11**] and its subsequent generalisations by Ge and al. [**20**], and Weth and al. [**6**] (for $k \geq 1$) the authors work with a smooth domain in $\mathbb{R}^n$ and assume that it has a small "hole". In the context of a compact manifold, this assumption is not enough: indeed, the entire compact manifold minus a small hole might retract to a point. In section 7 of Chapter 2, we show that, in the case of the canonical sphere the existence of a hole is not sufficient to get solutions to equation (1.11), showing that the hypothesis of Theorem 1.4 is necessary.

One can also let (M,g) to be a smooth, compact Riemannian manifold of dimension n with boundary. By this we understand that $\overline{M}$ is a compact, oriented submanifold of $(\tilde{M}, g)$ which is itself a smooth, compact Riemannian manifold without boundary and with the same metric g and dimension n. As one checks, this includes smooth bounded domains of $\mathbb{R}^n$. When the boundary $\partial M \neq \emptyset$, we let ν be its outward oriented normal vector in $\tilde{M}$. Then in addition to equation (1.2) one can also consider the following general boundary value problem on M

$$\begin{cases} Pu = f\,|u|^{2^\sharp_k - 2} u & \text{in } M \\ \partial^\alpha_\nu u = 0 & \text{on } \partial M \quad \text{for } |\alpha| \leq k-1. \end{cases} \tag{1.11}$$

where $f \in C^{0,\theta}(M)$ is a Hölder continuous function.

The Hilbert space $H_{k,0}^2(M)$ is similarly defined as the completion of the space $C_c^\infty(M)$ with respect to the norm $\|\cdot\|_{H_k^2}^2$ as defined earlier. We say that $u \in H_{k,0}^2(M)$ is a weak solution of equation (1.11) if

$$\int_M \Delta_g^{k/2} u, \Delta_g^{k/2}\varphi \, dv_g + \sum_{l=0}^{k-1} \int_M A_l(\nabla^l u \nabla^l \varphi) \, dv_g = \int_M |u|^{2_k^\sharp - 2} u\varphi \, dv_g$$

for all $\varphi \in H_{k,0}^2(M)$. The functional J_P is well defined on $H_{k,0}^2(M) \setminus \{0\}$ and its critical points corresponds to weak solutions of (1.11). Any weak solution to equation (1.11) is again a classical solution(see Chapter 2: Regularity) .
One can also consider the free functional

$$I_P(u) := \frac{1}{2} \int_M uP(u) \, dv_g - \frac{1}{2_k^\sharp} \int_M |u|^{2_k^\sharp} \, dv_g$$

on $H_{k,0}^2(M)$. Critical points $u \in H_{k,0}^2(M)$ of I_P are again weak solutions to equation (1.11).

Definition 1.0.1. *Let $(X, \|\cdot\|)$ be a Banach space and $F \in C^1(X)$. A sequence (u_m) in X is said to be a Palais-Smale sequence for F if $(F(u_m))_m$ has a limit in $\mathbb{R}$ when $m \to +\infty$, while $DF(u_m) \to 0$ strongly in X' as $m \to +\infty$.*

In [**38**] we describe the lack of relative compactness of Palais-Smale sequences for I_P, which is due to the noncompact embedding $H_{k,0}^2(M) \hookrightarrow L^{2_k^\sharp}(M)$. We obtain a characterization of the Palais-Smale sequences for I_P as a sum of bubbles plus a critical point of I_P (which can be trivial), a result in the spirit of Struwe's celebrated 1984 result. We consider Riemannian manifolds with or without boundary. The main idea is that often a non-convergent Palais-Smale sequence (u_m) or a blown up version of it splits up into a piece that converges weakly to a solution of the original problem u_0 and another one that converges to solutions of a closely related limiting problem. This is a very powerful result which is often used to show the existence of solutions to a variational problem.

In our case, because of the higher order of the polyharmonic operator, unlike the classical case of the Laplace operator ($k = 1$), in general there might be bubbles approaching the boundary of the domain, and this generates special type of bubbles which are solutions to the rescaled equation in the half space.

For Ω any open domain of $\mathbb{R}^n$, we let $\mathcal{D}_k^2(\Omega)$ be the completion of $C_c^\infty(\Omega)$ for the norm $u \mapsto \|\Delta^{k/2}u\|_2$. The limiting equations of (1.11) are

$$\Delta^k u = |u|^{2_k^\sharp - 2} u \text{ in } \mathbb{R}^n, \; u \in \mathcal{D}_k^2(\mathbb{R}^n) \tag{1.12}$$

$$\left\{ \begin{array}{ll} \Delta^k u = |u|^{2_k^\sharp - 2} u & \text{in } \mathbb{R}^n_- \\ \partial_\nu^\alpha u = 0 & \text{on } \partial\mathbb{R}^n_- \end{array} \right\}, \; u \in \mathcal{D}_k^2(\mathbb{R}^n_-) \tag{1.13}$$

where $\Delta := \Delta_{\text{Eucl}}$ is the Laplacian on $\mathbb{R}^n$ (with the minus sign convention) endowed with the Euclidean metric Eucl. Associated to the functional I_P is the limiting functional

$$E(u) := \frac{1}{2} \int_{\mathbb{R}^n} (\Delta^{k/2}u)^2 \, dx - \frac{1}{2_k^\sharp} \int_{\mathbb{R}^n} |u|^{2_k^\sharp} \, dx \text{ for all } u \in \mathcal{D}_k^2(\mathbb{R}^n).$$

The full H_k^2-decomposition of the Palais-Smale sequences for the functional I_P is given by the following theorem

Theorem 1.5. (*Mazumdar* [**38**], *see Chapter* 3) *Let* (u_m) *be a Palais-Smale sequence for the functional* I_P *on the space* $H_{k,0}^2(M)$. *Then there exists* $d \in \mathbb{N}$ *bubbles* $[(x_m^{(j)}), (r_m^{(j)}), u^{(j)}]$, $j = 1, ..., d$, *there exists* $u_\infty \in H_{k,0}^2(M)$ *a solution to* (1.11) *such that, up to a subsequence,*

$$u_m = u_\infty + \sum_{j=1}^{d} B_{x_m^{(j)}, r_m^{(j)}}(u^{(j)}) + o(1) \text{ where } \lim_{m \to +\infty} o(1) = 0 \text{ in } H_{k,0}^2(M)$$

and

$$I_P(u_m) = I_P(u_\infty) + \sum_{j=1}^{d} E(u^{(j)}) + o(1) \quad \text{as } m \to +\infty.$$

for definition of bubbles see: section 2 of Chapter 3. As one checks, for any nontrivial weak solution $u \in \mathcal{D}_k^2(\mathbb{R}^n)$ of (1.12) or (1.13)

$$E(u) \geq \beta^\sharp := \frac{k}{n} K_0(n,k)^{-n/2k} \tag{1.14}$$

When the Palais-Smale sequence is nonnegative, the bubbles are indeed positive and correspond to positive solutions of (1.12). We then have:

Theorem 1.6. (*Mazumdar* [**38**], *see Chapter* 3) *Let* (u_m) *be a Palais-Smale sequence for the functional* I_p *on the space* $H_{k,0}^2(M)$. *We assume that* $u_m \geq 0$ *for all* $m \in \mathbb{N}$. *Then there exists* $u_\infty \in H_{k,0}^2(M)$ *a solution to* (1.11), *there exists* $d \in \mathbb{N}$ *sequences* : $(x_m^{(1)}), \dots, (x_m^{(d)}) \in M$, $(r_m^{(1)}), \dots, (r_m^{(d)}) \in (0, +\infty)$ *such that* $r_m^{(j)} \to 0$ *and* $r_m^{(j)} = o(d(x_m^{(j)}, \partial M))$ *as* $m \to +\infty$ *for all* $j = 1, ..., d$, *and up to a subsequence,*

$$u_m = u_\infty + \sum_{j=1}^{d} \eta \left((\tilde{r}_m^{(j)})^{-1} exp_{x_m^{(j)}}^{-1}(\cdot) \right) \alpha_{n,k} \left(\frac{r_m^{(j)}}{(r_m^{(j)})^2 + d_g(\cdot, x_m^{(j)})^2} \right)^{\frac{n-2k}{2}} + o(1)$$

where $\lim_{m \to +\infty} o(1) = 0$ *in* $H_{k,0}^2(M)$, *and* η *is a smooth cut-off function and* $\tilde{r}_m^{(j)}$*'s are such that for all* $j = 1, ..., d$

$$\lim_{\alpha \to +\infty} \frac{r_\alpha^j}{\tilde{r}_\alpha^j} = 0 \text{ and } \tilde{r}_\alpha^j < \frac{d_g(x_\alpha^j, \partial M)}{2}$$

Moreover,

$$I_P(u_m) = I(u_\infty) + d\beta^\sharp + o(1) \quad \text{as } m \to +\infty$$

where $\beta^\sharp$ *is as in* (1.14).

When $k = 1$ and M is a smooth bounded domain of $\mathbb{R}^n$, Theorem 1.5 is the pioneering result of Struwe [**46**]. There have been several extensions. Without being exhaustive, we refer to Hebey-Robert [**33**] for $k = 2$ and manifolds without boundary, Saintier [**45**] for the $p-$Laplace operator, El-Hamidi-Vétois [**29**] for anisotropic operators and Almaraz [**2**] for nonlinear boundary conditions. When the manifold is the entire flat space $\mathbb{R}^n$, the decomposition is in the monograph by Fieseler-Tintarev [**48**].

Palais-Smale sequence are produced via critical point techniques, like the Mountain-Pass Lemma of Ambrosetti-Rabinowitz [**3**] or other topological methods (see for instance the monograph Ghoussoub [**21**] and the references therein).

For general higher-order problems, we also refer to Bartsch-Weth-Willem [**6**], Pucci-Serrin [**41**], Ge-Wei-Zhou [**20**], the general monograph Gazzola-Grunau-Sweers [**19**] and the references therein.

These works are the object of my following two papers (submitted)

[**37**] *GJMS-type Operators on a compact Riemannian manifold: Best constants and Coron-type solutions.* See Chapter 2 of this memoir.

[**38**] *Struwe's decomposition for a Polyharmonic Operator on a compact Riemannian manifold with or without boundary.* See Chapter 3 of this memoir.

Part 2

Let Ω be a bounded smooth oriented domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$. We define the Sobolev space $H^2_{1,0}(\Omega)$ as the completion of the space $C^\infty_c(\Omega)$, the space of compactly supported smooth functions in Ω, with respect to the norm

$$\|u\|^2_{H^2_{1,0}(\Omega)} = \int_\Omega |\nabla u|^2 \, dx$$

We let $2^* := \frac{2n}{n-2}$ be the critical Sobolev exponent for the embeding $H^2_{1,0}(\Omega) \hookrightarrow L^p(\Omega)$. Namely, the embedding is defined and continuous for $1 \leq p \leq 2^*$, and it is compact iff $1 \leq p < 2^*$. Let $a \in C^1(\overline{\Omega})$ be such that the operator $\Delta + a$ is coercive in Ω, that is, there exists a constant $A_0 > 0$ such that for all $\varphi \in H^2_{1,0}(\Omega)$

$$\int_\Omega \left(|\nabla\varphi|^2 + a\varphi^2\right) \, dx \geq A_0 \int_\Omega \varphi^2 \, dx \tag{1.15}$$

Solutions $u \in C^2(\overline{\Omega})$ to the problem

$$\begin{cases} \Delta u + a(x)u = u^{2^*-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.16}$$

(often referred to as "Brezis-Nirenberg problem") are critical points of the functional

$$u \mapsto \frac{\int_\Omega \left(\, |\nabla u|^2 + au^2 \, \right) dx}{\left(\int_\Omega |u|^{2^*} \, dx \right)^{2/2^*}},$$

and a natural way to obtain such critical points is to find minimizers to this functional, that is to prove that

$$\mu_a(\Omega) = \inf_{u \in H^2_{1,0}(\Omega)\setminus\{0\}} \frac{\int_\Omega \left(\, |\nabla u|^2 + au^2 \, \right) dx}{\left(\int_\Omega |u|^{2^*} \, dx \right)^{2/2^*}} \tag{1.17}$$

is achieved. There is a huge and extensive litterature on this problem, starting with the pioneering article of Brezis-Nirenberg [**10**] in which the authors completely solved the question of existence of minimizers for $\mu_a(\Omega)$ when a is a constant and $n \geq 4$ for any domain, and $n = 3$ for a ball. Their analysis took inspiration from the contributions of Aubin [**4**] in the resolution of the Yamabe problem. The case when a is arbitrary and $n = 3$ was solved by Druet [**13**] using blowup analysis.

In [**25**] and [**24**], Ghoussoub-Yuan and Ghoussoub-Kang suggested to approach the minimisation problem by adding a singularity in the equation as follows. For any $s \in [0, 2)$, we define

$$2^*(s) := \frac{2(n-s)}{n-2}$$

so that $2^* = 2^*(0)$. Weak solutions $u \in H^2_{1,0}(\Omega)\backslash\{0\}$ to the problem

$$\begin{cases} \Delta u + a(x)u = \frac{u^{2^*(s)-1}}{|x|^s} & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Note here that $0 \in \partial\Omega$ is a boundary point. Such solutions can be achieved as minimizers for the problem

$$\mu_{s,a}(\Omega) = \inf_{u \in H^2_{1,0}(\Omega)\backslash\{0\}} \frac{\int\limits_\Omega \left(|\nabla u|^2 + au^2 \right) dx}{\left(\int\limits_\Omega \frac{|u|^{2^*(s)}}{|x|^s}\, dx \right)^{2/2^*(s)}} \qquad \text{for } s \in (0,2) \tag{1.18}$$

Consider a sequence of positive real numbers $(s_\epsilon)_{\epsilon>0}$ such that $\lim\limits_{\epsilon\to 0} s_\epsilon = 0$. We let $(u_\epsilon)_{\epsilon>0} \in C^2\left(\overline{\Omega}\backslash\{0\}\right) \cap C^1\left(\overline{\Omega}\right)$ such that

$$\begin{cases} \Delta u_\epsilon + au_\epsilon = \frac{u_\epsilon^{2^*(s_\epsilon)-1}}{|x|^{s_\epsilon}} & \text{in } \Omega, \\ u_\epsilon > 0 & \text{in } \Omega, \\ u_\epsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.19}$$

Moreover, we assume that the (u_ϵ)'s are of minimal energy type in the sense that

$$\frac{\int\limits_\Omega \left(|\nabla u_\epsilon|^2 + au_\epsilon^2 \right) dx}{\left(\int\limits_\Omega \frac{|u_\epsilon|^{2^*(s_\epsilon)}}{|x|^s}\, dx \right)^{2/2^*(s_\epsilon)}} = \mu_{s_\epsilon,a}(\Omega) \leq \frac{1}{K(n,0)} + o(1) \tag{1.20}$$

as $\epsilon \to 0$, where $K(n, 0) > 0$ is the best constant in the Sobolev embedding which can be characterised as

$$\frac{1}{K(n,0)} = \inf_{u \in \mathscr{D}^{1,2}(\mathbb{R}^n)\backslash\{0\}} \frac{\int\limits_{\mathbb{R}^n} |\nabla u|^2\, dx}{\left(\int\limits_{\mathbb{R}^n} |u|^{2^*}\, dx \right)^{2/2^*}} \tag{1.21}$$

Indeed, it follows from Ghoussoub-Robert [**22, 23**] that such a family $(u_\epsilon)_\epsilon$ exists if the the mean curvature of $\partial\Omega$ at 0 is negative.

The lack of compactness of the critical Sobolev embeddings potentially generates a noncompactness of families of solutions to equations like (1.16). When a family is not relatively compact, we say that it is a *blow up sequence*. In the past years, there has been a considerable abundance of descriptions of blowing-up sequence, starting with the description in the sense of measures by Lions [**36**] and the description of

Palais-Smale sequence by Struwe [**46**] that we have discussed in the first part of this memoir. Other classical references for the blow-up analysis of nonlinear critical elliptic pdes are Rey [**42**], Adimurthi- Pacella-Yadava [**1**], Druet-Robert-Wei [**15**], Han [**30**], Hebey-Vaugon [**34**] and Khuri-Marques-Schoen [**35**]. In particular, for sequences of solutions, the optimal pointwise control of blow up is in Druet-Hebey-Robert [**14**]. The analysis of the 3D problem by Druet [**13**] and the monograph [**14**] by Druet-Hebey-Robert were important sources of inspiration.

Here, we are interested in studying the asymptotic behavior of the sequence $(u_\epsilon)_{\epsilon>0}$ as $\epsilon \to 0$. As proved in Proposition 3.2 of [**39**], if the weak limit u_0 of $(u_\epsilon)_\epsilon$ in $H^2_{1,0}(\Omega)$ is nontrivial, then the convergence is indeed strong and u_0 is a minimizer of $\mu_a(\Omega)$. In the spirit of the C^0-theory of Druet-Hebey-Robert [**14**], our first result is the following:

Theorem 1.7. (*Mazumdar* [**39**], *see Chapter* 4) *Let Ω be a bounded smooth oriented domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$, and let $a \in C^1(\overline{\Omega})$ be such that the operator $\Delta + a$ is coercive in Ω. Let $(s_\epsilon)_{\epsilon>0} \in (0,2)$ be a sequence such that $\lim\limits_{\epsilon\to 0} s_\epsilon = 0$. Suppose that the sequence $(u_\epsilon)_{\epsilon>0} \in H^2_{1,0}(\Omega)$, where for each $\epsilon > 0$, u_ϵ satisfies* (1.19) *and* (1.20), *is a* blowup sequence, *i.e*

$$u_\epsilon \rightharpoonup 0 \qquad \textit{weakly in } H^2_{1,0}(\Omega) \qquad \textit{as } \ \epsilon \to 0$$

We rescale and define

$$v_\epsilon(x) = \frac{u_\epsilon(x_\epsilon + k_\epsilon x)}{u_\epsilon(x_\epsilon)} \qquad \textit{for } x \in \frac{\Omega - x_\epsilon}{k_\epsilon}$$

where

$$k_\epsilon = |x_\epsilon|^{\frac{s_\epsilon}{2}} \mu_\epsilon^{\frac{2-s_\epsilon}{2}}$$

and

$$\mu_\epsilon^{-\frac{n-2}{2}} = u_\epsilon(x_\epsilon) = \max_{x\in\Omega} u_\epsilon(x).$$

Then there exists $v \in C^\infty(\mathbb{R}^n)$ such that $v \neq 0$ and for any $\eta \in C^\infty_c(\mathbb{R}^n)$

$$\eta v_\epsilon \rightharpoonup \eta v \qquad \textit{weakly in } H^2_1(\mathbb{R}^n) \qquad \textit{as } \epsilon \to 0$$

and

$$v_\epsilon \longrightarrow v \qquad \textit{in } C^1_{loc}(\mathbb{R}^n) \qquad \textit{as } \epsilon \to 0$$

Further $v(0) = 1$ and it satisfies the equation

$$\begin{cases} \Delta v = v^{2^*-1} & \textit{in } \mathbb{R}^n \\ v \geq 0 & \textit{in } \mathbb{R}^n \end{cases}$$

Next we obtain strong pointwise control

Theorem 1.8. (*Mazumdar* [**39**], *see Chapter* 4) *Let Ω be a bounded smooth oriented domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$, and let $a \in C^1(\overline{\Omega})$ be such that the operator $\Delta + a$ is coercive in Ω. Let $(s_\epsilon)_{\epsilon>0} \in (0,2)$ be a sequence such that $\lim\limits_{\epsilon\to 0} s_\epsilon = 0$. Suppose that the sequence $(u_\epsilon)_{\epsilon>0} \in H^2_{1,0}(\Omega)$, where for each $\epsilon > 0$, u_ϵ satisfies* (1.19) *and* (1.20), *is a* blowup sequence, *i.e*

$$u_\epsilon \rightharpoonup 0 \qquad \textit{weakly in } H^2_{1,0}(\Omega) \qquad \textit{as } \ \epsilon \to 0$$

Then, there exists $C > 0$ such that for all $\epsilon > 0$

$$u_\epsilon(x) \le C \left(\frac{\mu_\epsilon}{\mu_\epsilon^2 + |x - x_\epsilon|^2} \right)^{\frac{n-2}{2}} \qquad \text{for all } x \in \Omega$$

where

$$\mu_\epsilon^{-\frac{n-2}{2}} = u_\epsilon(x_\epsilon) = \max_{x\in\Omega} u_\epsilon(x).$$

Theorem 1.8 asserts that the pointwise control is the same as the control of the classical problem with $s_\epsilon = 0$: however, to prove this result, we need to perform a very delicate analysis of the blowup with the perturbation $s_\epsilon > 0$.

With this optimal pointwise control, we are able to obtain more informations on the localization of the blowup point $x_0 := \lim_{\epsilon\to 0} x_\epsilon$ and the blowup parameter $(\mu_\epsilon)_\epsilon$. We let $G^a : \overline{\Omega} \times \overline{\Omega} \setminus \{(x,x) : x \in \overline{\Omega}\} \longrightarrow \mathbb{R}$ is the Green's function of the coercive operator $\Delta + a$ in Ω with Dirichlet boundary conditions. For any $x \in \Omega$ we write G_x^a as:

$$G_x^a(y) = \frac{1}{(n-2)\omega_{n-1}|x-y|^{n-2}} + g_x^a(y)$$

where ω_{n-1} is the area of the $(n-1)$- sphere. In dimension $n = 3$ or when $a \equiv 0$, one has that $g_x^a \in C^2(\overline{\Omega} \setminus \{x\}) \cap C^{0,\theta}(\Omega)$ for some $0 < \theta < 1$, and g^a is called the regular part of the Green's function G^a. In particular, when $n = 3$ or $a \equiv 0$, $m_x(\Omega, a) := g_x^a(x)$ is defined for all $x \in \Omega$ and is called the mass of the operator $\Delta + a$.

Theorem 1.9. (*Mazumdar* [**39**], *see Chapter* 4) *Let Ω be a bounded smooth oriented domain of $\mathbb{R}^n$, $n \ge 3$, such that $0 \in \partial\Omega$, and let $a \in C^1(\overline{\Omega})$ be such that the operator $\Delta + a$ is coercive in Ω. Let $(s_\epsilon)_{\epsilon>0} \in (0,2)$ be a sequence such that $\lim\limits_{\epsilon\to 0} s_\epsilon = 0$. Suppose that the sequence $(u_\epsilon)_{\epsilon>0} \in H_{1,0}^2(\Omega)$, where for each $\epsilon > 0$, u_ϵ satisfies* (1.19) *and* (1.20), *is a* blowup sequence, *i.e*

$$u_\epsilon \rightharpoonup 0 \qquad \text{weakly in } H_{1,0}^2(\Omega) \qquad \text{as } \epsilon \to 0$$

We let $(\mu_\epsilon)_\epsilon \in (0, +\infty)$ and $(x_\epsilon)_\epsilon \in \Omega$ be such that

$$\mu_\epsilon^{-\frac{n-2}{2}} = u_\epsilon(x_\epsilon) = \max_{x\in\Omega} u_\epsilon(x).$$

We define $x_0 := \lim_{\epsilon\to 0} x_\epsilon$.

Suppose

$$x_0 \in \Omega \text{ is an interior point.}$$

Then

$$\lim_{\epsilon\to 0} \frac{s_\epsilon}{\mu_\epsilon^2} = 2^* K(n,0)^{\frac{2^*}{2^*-2}} d_n \; a(x_0) \qquad \text{for } n \ge 5$$

$$\lim_{\epsilon\to 0} \frac{s_\epsilon}{\mu_\epsilon^2 \log(1/\mu_\epsilon)} = 256\omega_3 K(4,0)^2 \; a(x_0) \qquad \text{for } n = 4$$

$$\lim_{\epsilon\to 0} \frac{s_\epsilon}{\mu_\epsilon^{n-2}} = -nb_n^2 K(n,0)^{n/2} g_{x_0}^a(x_0) \qquad \text{for } n = 3 \text{ or } a \equiv 0.$$

where $g^a_{x_0}(x_0)$ *the mass at the point* $x_0 \in \Omega$ *for the operator* $\Delta + a$,

$$d_n = \int\limits_{\mathbb{R}^n} \frac{1}{\left(1 + \frac{|x|^2}{n(n-2)}\right)^{n-2}}\, dx \quad \text{for } n \geq 5 \ ; \ b_n = \int\limits_{\mathbb{R}^n} \frac{1}{\left(1 + \frac{|x|^2}{n(n-2)}\right)^{\frac{n+2}{2}}}\, dx$$

and ω_3 *is the area of the* 3*- sphere.*

Suppose if

$$\lim_{\epsilon \to 0} x_\epsilon = x_0 \in \partial\Omega.$$

When $n = 3$ *or* $a \equiv 0$*, then as* $\epsilon \to 0$

$$\lim_{\epsilon \to 0} \frac{s_\epsilon d(x_\epsilon, \partial\Omega)^{n-2}}{\mu_\epsilon^{n-2}} = \frac{n^{n-1}(n-2)^{n-1} K(n,0)^{n/2} \omega_{n-1}}{2^{n-2}}.$$

Moreover, $d(x_\epsilon, \partial\Omega) = (1 + o(1))|x_\epsilon|$ *as* $\epsilon \to 0$*. In particular* $x_0 = 0$*.*

Indeed, we also tackle the general case $n \geq 4$ or $a \not\equiv 0$. The detailed results are in Theorems 4.3 and 4.10 of Chapter 4.

The main difficulty in our analysis is due to the natural singularity at $0 \in \partial\Omega$. Indeed, there is a balance between two facts. First, since $s_\epsilon > 0$, this singularity exists and has an influence on the analysis, and in particular on the Pohozaev identity (for details see the statement of Theorem 1.9 above). But, second, since $s_\epsilon \to 0$, the singularity should cancel, at least asymptotically. In this perspective, our results are twofolds.

The influence and the role of $s_\epsilon > 0$ is much more striking. Compared to the case $s_\epsilon = 0$, there is an additional term in the Pohozaev identity involving s_ϵ. Heuristically, this is due to the fact that the limiting equation $\Delta u = |x|^{-s} u^{2^*(s)-1}$ is not invariant under the action of the translations when $s > 0$.

This part is the subject of my work:

[**39**] Blow-up Analysis For a Sequence of Solutions of The Critical Hardy-Sobolev Equations. See Chapter 4 of this memoir.

Regarding notation, we tried to make it as much unified as possible. Nevertheless, the main specific notation will be introduced chapter by chapter.

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Part 1

Polyharmonic operators on Riemannian manifolds

CHAPTER 2

GJMS-type Operators on a compact Riemannian manifold: Best constants and Coron-type solutions

ABSTRACT. In this chapter we investigate the existence of solutions to a nonlinear elliptic problem involving critical Sobolev exponent for a polyharmomic operator on a Riemannian manifold M. We first show that the best constant of the Sobolev embedding on a manifold can be chosen as close as one wants to the Euclidean one, and as a consequence derive the existence of minimizers when the energy functional goes below a quantified threshold. Next, higher energy solutions are obtained by Coron's topological method, provided that the minimizing solution does not exist. To perform this topological argument, we overcome the difficulty of dealing with polyharmonic operators on a Riemannian manifold and adapting Lions's concentration-compactness lemma. Unlike Coron's original argument for a bounded domain in $\mathbb{R}^n$, we need to do more than chopping out a small ball from the manifold M. Indeed, our topological assumption that a small sphere on M centred at a point $p \in M$ does not retract to a point in $M\backslash\{p\}$ is necessary, as shown for the case of the canonical sphere where chopping out a small ball is not enough.

2.1. Introduction

Let M be a compact manifold of dimension $n \geq 3$ without boundary. Let k be a positive integer such that $2k < n$. Taking inspiration from the construction of the ambient metric of Fefferman-Graham [**15**] (see [**16**] for an extended analysis of the ambient metric), Graham-Jenne-Mason-Sparling [**19**] have defined a family of conformally invariant operators defined for any Riemannian metric. More precisely, for any Riemannian metric g on M, there exists a local differential operator P_g : $C^\infty(M) \to C^\infty(M)$ such that $P_g = \Delta_g^k + lot$ where $\Delta_g := -\mathrm{div}_g(\nabla)$, and, given $u \in C^\infty(M)$ and defining $\hat{g} = u^{\frac{4}{n-2k}} g$, we have that

$$P_{\hat{g}}(\varphi) = u^{-\frac{n+2k}{n-2k}} P_g(u\varphi) \text{ for all } \varphi \in C^\infty(M). \tag{2.1}$$

Moreover, P_g is self-adjoint with respect to the L^2-scalar product. A scalar invariant is associated to this operator, namely the $Q-$curvature, denoted as $Q_g \in C^\infty(M)$. When $k = 1$, P_g is the conformal Laplacian and the $Q-$curvature is the scalar curvature multiplied by a constant. When $k = 2$, P_g is the Paneitz operator introduced in [**29**]. The $Q-$curvature was introduced by Branson and Ørsted [**10**]. The definition of Q_g was then generalized by Branson [**8, 9**]. In the specific case $n > 2k$, we have that $Q_g := \frac{2}{n-2k} P_g(1)$. Then, taking $\varphi \equiv 1$ in (2.1), we get that $P_g u = \frac{n-2k}{2} Q_{\hat{g}} u^{\frac{n+2k}{n-2k}}$ on M. Therefore, prescribing the $Q-$curvature in a conformal class amounts to solving a nonlinear elliptic partial differential equation(PDE)of $2k^{th}$ order. Results for the prescription of the $Q-$curvature problem for the Paneitz operator (namely $k=2$) are in Djadli-Hebey-Ledoux [**13**], Robert [**31**], Esposito-Robert [**14**]. Recently, Gursky-Malchiodi [**20**] proved the existence of a metric with constant $Q-$curvature (still for $k=2$) provided certain geometric hypotheses on the manifold (M,g) holds. These hypotheses have been simplified by Hang-Yang [**21**] (see the lecture notes [**22**])

In the present chapter, we are interested in a generalization of the prescription of the $Q-$curvature problem. Namely, given $f\in C^\infty(M)$, we investigate the existence of $u\in C^\infty(M)$, $u>0$, such that

$$Pu = fu^{2_k^\sharp -1} \text{ in } M, \tag{2.2}$$

where $2_k^\sharp := \frac{2n}{n-2k}$ and $P: C^\infty(M)\to C^\infty(M)$ is a smooth self-adjoint $2k^{th}$ order partial differential operator defined by

$$Pu = \Delta_g^k u + \sum_{l=0}^{k-1}(-1)^l \nabla^{j_l\dots j_1}\left(A_l(g)_{i_1\dots i_l, j_1\dots j_l}\nabla^{i_1\dots i_l}u\right) \tag{2.3}$$

where the indices are raised via the musical isomorphism and for all $l\in\{0,\dots,k-1\}$, $A_l(g)$ is a smooth symmetric T^0_{2l}-tensor field on M (that is: $A_l(g)(X,Y) = A_l(g)(Y,X)$ for all T^l_0-tensors X,Y on M). When $P:=P_g$, then (2.2) is equivalent to say that $Q_{\hat g} = \frac{2}{n-2k}f$ with $\hat g = u^{\frac{4}{n-2k}}g$.

The conformal invariance (2.1) of the geometric operator P_g yields obstruction to the existence of solutions to (2.2). The historical reference here is Kazdan-Warner [**25**]; for the general GJMS operators, we refer to Delanoë-Robert [**12**]. In particular, it follows from [**12**] that on the canonical sphere $(\mathbb{S}^n, \text{can})$, there is no positive solution $u\in C^\infty(\mathbb{S}^n)$ to $P_{\text{can}}u = (1+\epsilon\varphi)u^{2_k^\sharp-1}$ for all $\epsilon\neq 0$ and all first spherical harmonic φ. For the conformal Laplacian (that is $k=1$), Aubin [**3**] proved that the existence of solutions is guaranteed if a functional goes below a specific threshold. We generalize this result for any $k\geq 1$ in Theorem 2.3. In the case of a smooth bounded domain, Coron [**11**] introduced a variational method based on topological arguments, provided the minimizing solution does not exist. Our main theorem is in this spirit:

Theorem 2.1. *Let (M,g) be a smooth, compact Riemannian manifold of dimension n and let k be a positive integer such that $2k<n$. We let P be a coercive operator as in* (2.3)*. Let $\iota_g>0$ be the injectivity radius of the manifold M. Suppose that the manifold M contains a point x_0 such that the embedded $(n-1)-$dimensional sphere $\mathbb{S}_{x_0}(\iota_g/2) := \{x\in M/\, d_g(x,x_0)=\iota_g/2\}$ is not contractible in $M\backslash\{x_0\}$. Then there exists $\epsilon_0\in(0,\frac{\iota_g}{2})$ such that the equation*

$$\begin{cases} Pu = |u|^{2_k^\sharp-2}u & \text{in } \Omega_M \\ D^\alpha u = 0 & \text{on } \partial\Omega_M \quad \text{for } |\alpha|\leq k-1 \end{cases} \tag{2.4}$$

has a non-trivial $C^{2k}(\Omega_M)$ solution for $\Omega_M := M\backslash \overline{B}_{x_0}(\epsilon_0)$. Moreover, if the Green's Kernel of P on Ω_M is positive, then we can choose $u>0$.

In the original result of Coron [**11**] (see also Weth and al. [**6**] for the case $k=2$), the authors work with a smooth domain of $\mathbb{R}^n$ and assume that it has a small "hole". In the context of a compact manifold, this assumption is not enough: indeed, the entire compact manifold minus a small hole might retract on a point.

We discuss the example of the canonical sphere in Section 2.7, where the existence of a hole is not sufficient to get solutions to (2.2).
Concerning higher-order problems, we refer to Bartsch-Weth-Willem [**6**], Pucci-Serrin [**30**], Ge-Wei-Zhou [**18**], the general monograph Gazzola-Grunau-Sweers [**17**] and the references therein.

Among other tools, the proof of Theorem 2.1 uses a Lions-type Concentration Compactness Lemma adapted to the context of a Riemannian manifold: this will be the object of Theorem 2.4.

Equation (2.2) has a variational structure. Since P is self-adjoint in L^2, we have that for all $u, v \in C^\infty(M)$.

(2.5)
$$\int_M uP(v)\,dv_g = \int_M vP(u)\,dv_g = \int_M \Delta_g^{k/2} u \Delta_g^{k/2} v\,dv_g + \sum_{l=0}^{k-1} \int_M A_l(g)(\nabla^l u, \nabla^l v)\,dv_g$$

where
$$\Delta_g^{l/2} u := \begin{cases} \Delta_g^m u & \text{if } l = 2m \text{ is even} \\ \nabla \Delta_g^m u & \text{if } l = 2m+1 \text{ is odd} \end{cases}$$

and, when $l = 2m+1$ is odd, $\Delta_g^{k/2} u\, \Delta_g^{k/2} v = \left(\nabla \Delta_g^m u, \nabla \Delta_g^m v\right)_g$. If P is coercive and $f > 0$, then, up to multiplying by a constant, any solution $u \in C^\infty(M)$ to (2.2) is a critical point of the functional

$$u \mapsto J_P(u) := \frac{\int_M uP(u)\,dv_g}{\left(\int_M f|u|^{2_k^\sharp}\,dv_g\right)^{2/2_k^\sharp}} \tag{2.6}$$

It follows from (2.5) that J_P makes sense in the Sobolev spaces $H_k^2(M)$, where for $1 \le l \le k$, $H_l^2(M)$ which is the completion of $C^\infty(M)$ with respect to the $u \mapsto \sum_{\alpha=0}^{l} \|\nabla^\alpha u\|_2$. Equivalently (see Robert [**32**]), $H_l^2(M)$ is also the completion of the space $C^\infty(M)$ with respect to the norm

$$\|u\|_{H_l^2}^2 := \sum_{\alpha=0}^{l} \int_M (\Delta_g^{\alpha/2} u)^2\,dv_g. \tag{2.7}$$

By the Sobolev embedding theorem we get a continuous but not compact embedding of $H_k^2(M)$ into $L^{2_k^\sharp}(M)$. The continuity of the embedding $H_k^2(M) \hookrightarrow L^{2_k^\sharp}(M)$ yields a pair of real numbers A, B such that for all $u \in H_k^2(M)$

$$\|u\|_{L^{2_k^\sharp}}^2 \le A \int_M (\Delta_g^{k/2} u)^2\,dv_g + B\,\|u\|_{H_{k-1}^2}^2 \tag{2.8}$$

See for example Aubin [**4**] or Hebey [**23**]. Following the terminology introduced by Hebey, we then define

(2.9)
$$\mathcal{A}(M) := \inf\{A \in \mathbb{R} : \exists\ B \in \mathbb{R} \text{ with the property that inequality (2.8) holds}\}$$

As for the classical case $k = 1$ (see Aubin [**4**]), the value of $\mathcal{A}(M)$ depends only on k and the dimension k. More precisely, we let $\mathscr{D}^{k,2}(\mathbb{R}^n)$ be the completion of $C_c^\infty(\mathbb{R}^n)$ for the norm $u \mapsto \|\Delta^{k/2}u\|_2$, and we define $K_0(n,k) > 0$

$$\frac{1}{K_0(n,k)} := \inf_{u\in\mathscr{D}^{k,2}(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n}(\Delta^{k/2}u)^2\,dx}{\left(\int_{\mathbb{R}^n}|u|^{2_k^\sharp}\,dx\right)^{\frac{2}{2_k^\sharp}}} \tag{2.10}$$

as the best constant in the Sobolev's continuous embedding $\mathscr{D}^{k,2}(\mathbb{R}^n) \hookrightarrow L^{2_k^\sharp}(\mathbb{R}^n)$. Our second result is the following:

Theorem 2.2. *Let (M,g) be a smooth, compact Riemannian manifold of dimension n and let k be a positive integer such that $2k < n$. Then $\mathcal{A}(M) = K_0(n,k) > 0$. In particular, for any $\epsilon > 0$, there exists $B_\epsilon \in \mathbb{R}$ such that for all $u \in H_k^2(M)$ one has*

$$\left(\int_M |u|^{2_k^\sharp}\,dv_g\right)^{\frac{2}{2_k^\sharp}} \le (K_0(n,k)+\epsilon)\int_M (\Delta_g^{k/2}u)^2\,dv_g + B_\epsilon\,\|u\|^2_{H^2_{k-1}} \tag{2.11}$$

As a consequence of this result, we will be able to prove the existence of solutions to (2.2) when the functional J_P goes below a quantified threshold, see Theorem 2.3.

This chapter is organized as follows. In Section 2.2, we study the best-constant problem and prove Theorem 2.2. In Section 2.3, we prove Theorem 2.3 by classical minimizing method. In Section 2.4, we prove a Concentration-Compactness Lemma in the spirit of Lions. Section 2.5 is devoted to test-functions estimates and the proof of the existence of solutions to (2.4) via a Coron-type topological method. Section 2.6 deals with positive solutions, and Section 2.7 with the necessity of the topological assumption of Theorem 2.1. The appendices concern regularity and a general comparison between geometric norms.

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2.2. The Best Constant

It follows from Lions [**26**] and Swanson [**34**] that the extremal functions for the Sobolev inequality (2.10) exist and are exactly multiples of the functions

$$U_{a,\lambda} = \alpha_{n,k}\left(\frac{\lambda}{1+\lambda^2|x-a|^2}\right)^{\frac{n-2k}{2}} \quad a\in\mathbb{R}^n, \lambda>0 \tag{2.12}$$

where the choice of $\alpha_{n,k}$'s are such that for all λ, $\|U_{a,\lambda}\|_{2_k^\sharp} = 1$ and $\|U_{a,\lambda}\|^2_{\mathscr{D}^{k,2}} = \frac{1}{K_0(n,k)}$. They satisfies the equation $\Delta^k u = \frac{1}{K_0(n,k)}|u|^{2_k^\sharp-2}u$ in $\mathbb{R}^n$

Next we consider the case of a compact Riemmanian manifold. The first result we have in this direction is the following.

Lemma 2.2.1. *Let (M,g) be a smooth, compact Riemannian manifold of dimension n and let k be a positve integer such that $2k < n$. Any constant A in inequality (2.8) has to be greater than or equal to $K_0(n,k)$, whatever the constant B be.*

Proof of Lemma 2.2.1*:* We fix $\epsilon > 0$ small. It follows from Lemma (2.9.1) that there exists, $\delta_0 \in (0, \iota_g)$ depending only on $(M, g), \epsilon$, where ι_g is the injectivity radius of M, such that for any point $p \in M$, any $0 < \delta < \delta_0$, $l \leq k$ and $u \in C_c^\infty(B_0(\delta))$

$$\int_M (\Delta_g^{l/2}(u \circ exp_p^{-1}))_g^2 \, dv_g \leq (1+\epsilon) \int_{\mathbb{R}^n} (\Delta^{l/2} u)^2 \, dx \tag{2.13}$$

and

$$(1-\epsilon) \left(\int_{\mathbb{R}^n} |u|^{2_k^\sharp} \, dx \right)^{2/2_k^\sharp} \leq \left(\int_M |u \circ exp_p^{-1}|^{2_k^\sharp} \, dv_g \right)^{2/2_k^\sharp} \tag{2.14}$$

Then plugging the above inequalities into (2.8) we obtain that any $u \in C_c^\infty(B_0(\delta))$ satisfies

$$\left(\int_{\mathbb{R}^n} |u|^{2_k^\sharp} \, dx \right)^{2/2_k^\sharp} \leq \frac{1+\epsilon}{1-\epsilon} A \int_{\mathbb{R}^n} (\Delta^{k/2} u)^2 \, dx + C_\epsilon \sum_{l=0}^{k-1} \int_{\mathbb{R}^n} |\nabla^l u|^2 \, dx \tag{2.15}$$

Let $v \in C_c^\infty(\mathbb{R}^n)$ with $supp(v) \subset B_0(R_0)$. For $\lambda > 1$ let $v_\lambda = v(\lambda x)$. Then for λ large, $supp(v_\lambda) \subset B_0(\delta)$. Taking $u \equiv v_\lambda$ in (2.15), a change of variable yields

$$\frac{1}{\lambda^{n-2k}} \left(\int_{\mathbb{R}^n} |v|^{2_k^\sharp} \, dx \right)^{2/2_k^\sharp} \leq \frac{1+\epsilon}{1-\epsilon} \cdot \frac{A}{\lambda^{n-2k}} \int_{\mathbb{R}^n} (\Delta^{k/2} v)^2 \, dx + C_\epsilon \sum_{l=0}^{k-1} \frac{1}{\lambda^{n-2l}} \int_{\mathbb{R}^n} |\nabla^l v|^2 \, dx \tag{2.16}$$

Multiplying by λ^{n-2k} and letting $\lambda \to +\infty$, we get that for all $v \in \mathscr{D}^{k,2}(\mathbb{R}^n)$, we have

$$\left(\int_{\mathbb{R}^n} |v|^{2_k^\sharp} \, dx \right)^{2/2_k^\sharp} \leq \frac{1+\epsilon}{1-\epsilon} A \int_{\mathbb{R}^n} (\Delta^{k/2} v)^2 \, dx \tag{2.17}$$

Therefore $\frac{1+\epsilon}{1-\epsilon} A \geq K_0(n,k)$ for all $\epsilon > 0$, and letting $\epsilon \to 0$ yields $A \geq K_0(n,k)$. This ends the proof of Lemma 2.2.1. □

We now prove (2.11) to get Theorem 2.2.

Step 1: A local inequality. From a result of Anderson (*Main lemma* 2.2 of [**2**]) it follows that for any point $p \in M$ there exists a harmonic coordinate chart φ around p. Then from Lemma 2.9.1, for any $0 < \epsilon < 1$, there exists $\tau > 0$ small enough such that for any point $p \in M$ and for any $u \in C_c^\infty(B_p(\tau))$, one has

$$\int_{\mathbb{R}^n} (\Delta^{k/2}(u \circ \varphi^{-1}))^2 \, dx \leq \left(1 + \frac{\epsilon}{3K_0(n,k)} \right) \int_M (\Delta_g^{k/2} u)^2 \, dv_g \tag{2.18}$$

and

$$\left(\int_M |u|^{2_k^\sharp} \, dv_g \right)^{2/2_k^\sharp} \leq \left(1 + \frac{\epsilon}{3K_0(n,k)} \right) \left(\int_{\mathbb{R}^n} |u \circ \varphi^{-1}|^{2_k^\sharp} \, dx \right)^{2/2_k^\sharp}. \tag{2.19}$$

The expression for the Laplacian Δ_g in the harmonic coordinates is $\Delta_g u = -g_{ij}\partial_{ij}u$. Then (2.10) implies that for any $u \in C_c^\infty(B_p(\tau))$

$$\left(\int_M |u|^{2_k^\sharp}\, dv_g\right)^{2/2_k^\sharp} \le (K_0(n,k)+\epsilon)\int_M (\Delta_g^{k/2}u)^2\, dv_g \tag{2.20}$$

Step 2: Finite covering and proof of the global inequality. Since M is compact, it can be covered by a finite number of balls $B_{p_i}(\tau/2)$, $i = 1, \ldots, N$. Let $\alpha_i \in C_c^\infty(B_{p_i}(\tau))$ be such that $0 \le \alpha_i \le 1$ and $\alpha_i = 1$ in $B_{p_i}(\tau/2)$. We set

$$\eta_i = \frac{\alpha_i^2}{\sum\limits_{j=1}^{N}\alpha_j^2} \tag{2.21}$$

Then $(\eta_i)_{i=1,\ldots,N}$ is a partition of unity subordinate to the cover $(B_{p_i}(\tau))_{i=1,\ldots,N}$ such that $\sqrt{\eta_i}$'s are smooth and $\sum\limits_{i=1}^{N}\eta_i = 1$. In the sequel, C denote any positive constant depending on k, n, the metric g on M and the functions $(\eta_i)_{i=1,\ldots,N}$. Now for any $u \in C^\infty(M)$, we have

$$\|u\|_{2_k^\sharp}^2 = \|u^2\|_{2_k^\sharp/2} = \left\|\sum_{i=1}^{N}\eta_i u^2\right\|_{2_k^\sharp/2} \le \sum_{i=1}^{N}\left\|\eta_i u^2\right\|_{2_k^\sharp/2} = \sum_{i=1}^{N}\|\sqrt{\eta_i}u\|_{2_k^\sharp}^2 \tag{2.22}$$

So for any $u \in C^\infty(M)$, using inequality (2.20) we obtain that

$$\left(\int_M |u|^{2_k^\sharp}\, dv_g\right)^{2/2_k^\sharp} \le (K_0(n,k)+\epsilon)\sum_{i=1}^{N}\int_M (\Delta_g^{k/2}(\sqrt{\eta_i}u))_g^2\, dv_g \tag{2.23}$$

Next we claim that there exists $C > 0$ such that

$$\sum_{i=1}^{N}\int_M (\Delta_g^{k/2}(\sqrt{\eta_i}u))^2\, dv_g \le \int_M (\Delta_g^{k/2}u)^2\, dv_g + C\,\|u\|_{H_{k-1}^2}^2 \tag{2.24}$$

Assuming that (2.24) holds we have from (2.23)

$$\left(\int_M |u|^{2_k^\sharp}\, dv_g\right)^{2/2_k^\sharp} \le (K_0(n,k)+\epsilon)\int_M (\Delta_g^{k/2}u)_g^2\, dv_g + (K_0(n,k)+\epsilon)C\,\|u\|_{H_{k-1}^2}^2 \tag{2.25}$$

this proves (2.11), and therefore, with Lemma 2.2.1, this proves Theorem 2.8. We are now left with proving (2.24).

Step 3: Proof of (2.24)**:** For any positive integer m, one can write that

$$\Delta_g^m(\sqrt{\eta_i}u) = \sqrt{\eta_i}\Delta_g^m u + \mathcal{P}_g^{(2m-1,1)}(u,\sqrt{\eta_i}) + \mathcal{L}_{\sqrt{\eta_i},g}^{2m-2}(u) \tag{2.26}$$

where

(2.27)

$$\mathcal{P}_g^{(2m-1,1)}(u,\sqrt{\eta_i}) = \sum_{|l|=2m-1,|\beta|=1} (a_{l,\beta}\partial_\beta\sqrt{\eta_i})\nabla^l u, \text{ and } \mathcal{L}^{2m-2}_{\sqrt{\eta_i},g}(u) = \sum_{|l|=0}^{2m-2} a_l(\sqrt{\eta_i})\ \nabla^l u$$

the coefficients $a_{l,\beta}$ and $a_l(\sqrt{\eta_i})$ are smooth functions on M. The $a_{l,\beta}$'s depends only on the metric g and on the manifold M and $a_l(\sqrt{\eta_i})$'s depends both on the metric g, the function $\sqrt{\eta_i}$ and its derivatives upto order $2m$. We shall use the same notations $\mathcal{P}_g^{(2m-1,1)}(u,\sqrt{\eta_i})$, $\mathcal{L}^{2m-2}_{\sqrt{\eta_i},g}(u)$ for any expression of the above form.

Step 3.1: ***k* is even.** We then write $k = 2m$, $m \geq 1$, and then

$$\begin{aligned}
&\sum_{i=1}^N \int_M \left(\Delta_g^m(\sqrt{\eta_i}u)\right)^2\ dv_g = \sum_{i=1}^N \int_M \eta_i \left(\Delta_g^m u\right)^2\ dv_g \\
&+\sum_{i=1}^N \int_M \left(\mathcal{P}_g^{(2m-1,1)}(u,\sqrt{\eta_i})\right)^2\ dv_g + \sum_{i=1}^N \int_M \left(\mathcal{L}^{2m-2}_{\sqrt{\eta_i},g}(u)\right)^2\ dv_g \\
&+2\sum_{i=1}^N \int_M \sqrt{\eta_i}\Delta_g^m u\ \mathcal{P}_g^{(2m-1,1)}(u,\sqrt{\eta_i})\ dv_g + 2\sum_{i=1}^N \int_M \sqrt{\eta_i}\Delta_g^m u\ \mathcal{L}^{2m-2}_{\sqrt{\eta_i},g}(u)\ dv_g \\
&+2\sum_{i=1}^N \int_M \mathcal{P}_g^{(2m-1,1)}(u,\sqrt{\eta_i})\ \mathcal{L}^{2m-2}_{\sqrt{\eta_i},g}(u)\ dv_g
\end{aligned} \tag{2.28}$$

We note that

(2.29)

$$\sum_{i=1}^N \int_M \left(\mathcal{P}_g^{(2m-1,1)}(u,\sqrt{\eta_i})\right)^2\ dv_g \leq C\,\|u\|^2_{H^2_{2m-1}}. \text{ and } \sum_{i=1}^N \int_M \left(\mathcal{L}^{2m-2}_{\sqrt{\eta_i},g}(u)\right)^2\ dv_g \leq C\,\|u\|^2_{H^2_{2m-2}}.$$

On the other hand

$$\begin{aligned}
&\sum_{i=1}^N \int_M \sqrt{\eta_i}\Delta_g^m u\ \mathcal{P}_g^{(2m-1,1)}(u,\sqrt{\eta_i})\ dv_g = \sum_{i=1}^N \sum_{|l|=2m-1}\sum_{|\beta|=1}\int_M (\sqrt{\eta_i}\Delta_g^m u)((a_{l,\beta}\partial_\beta\sqrt{\eta_i})\nabla^l u)\ dv_g \\
&= \frac{1}{2}\sum_{i=1}^N \sum_{|l|=2m-1}\sum_{|\beta|=1}\int_M (\Delta_g^m u)((a_{l,\beta}\partial_\beta\eta_i)\nabla^l u)\ dv_g \\
&= \frac{1}{2}\sum_{|l|=2m-1}\sum_{|\beta|=1}\int_M (\Delta_g^m u)((a_{l,\beta}\ \partial_\beta(\sum_{i=1}^N \eta_i))\nabla^l u)\ dv_g = 0
\end{aligned} \tag{2.30}$$

while using the integration by parts formula we obtain

$$\sum_{i=1}^N \int_M \sqrt{\eta_i}\Delta_g^m u\ \mathcal{L}^{2m-2}_{\sqrt{\eta_i},g}(u)\ dv_g \leq C\,\|u\|^2_{H^2_{2m-1}} \tag{2.31}$$

and by Hölder inequality

$$\sum_{i=1}^{N} \int_M \mathcal{P}_g^{(2m-1,1)}(u, \sqrt{\eta_i})\ \mathcal{L}_{\sqrt{\eta_i},g}^{2m-2}(u)\ dv_g \le C\, \|u\|_{H^2_{2m-1}}^2 \tag{2.32}$$

Hence if k is even, then

$$\sum_{i=1}^{N} \int\limits_M \left(\Delta_g^m(\sqrt{\eta_i}u)\right)^2\ dv_g \le \int\limits_M \left(\Delta_g^m u\right)^2\ dv_g + C\, \|u\|_{H^2_{2m-1}}^2 \tag{2.33}$$

So we have the claim for k even.

Step 3.2: k is odd. We then write $k = 2m+1$ with $m \ge 0$. We have

(2.34)

$$\nabla\left(\Delta_g^m(\sqrt{\eta_i}u)\right) = \sqrt{\eta_i}\ \nabla\left(\Delta_g^m u\right) + \left(\Delta_g^m u\right)\ \nabla\sqrt{\eta_i} + \nabla\left(\mathcal{P}_g^{(2m-1,1)}(u, \sqrt{\eta_i})\right) + \nabla\left(\mathcal{L}_{\sqrt{\eta_i},g}^{2m-2}(u)\right)$$

and so

(2.35)

$$\sum_{i=1}^{N} \int\limits_M \left|\nabla\left(\Delta_g^m(\sqrt{\eta_i}u)\right)\right|^2 dv_g = \sum_{i=1}^{N} \int\limits_M \eta_i \left|\nabla\left(\Delta_g^m u\right)\right|^2\ dv_g + \sum_{i=1}^{N} \int\limits_M (\Delta_g^m u)^2 \left|\nabla\sqrt{\eta_i}\right|^2\ dv_g$$

$$+ \sum_{i=1}^{N} \int\limits_M \left|\nabla\left(\mathcal{P}_g^{(2m-1,1)}(u, \sqrt{\eta_i})\right)\right|^2\ dv_g + \sum_{i=1}^{N} \int\limits_M \left|\nabla\left(\mathcal{L}_{\sqrt{\eta_i},g}^{2m-2}(u)\right)\right|^2\ dv_g$$

$$+ 2\sum_{i=1}^{N} \int\limits_M \left(\sqrt{\eta_i}\ \nabla\left(\Delta_g^m u\right), \left(\Delta_g^m u\right)\ \nabla\sqrt{\eta_i}\ \right) dv_g + 2\sum_{i=1}^{N} \int\limits_M \left(\sqrt{\eta_i}\ \nabla\left(\Delta_g^m u\right), \nabla\left(\mathcal{P}_g^{(2m-1,1)}(u, \sqrt{\eta_i})\right)\right) dv_g$$

$$+ 2\sum_{i=1}^{N} \int\limits_M \left(\sqrt{\eta_i}\ \nabla\left(\Delta_g^m u\right), \nabla\left(\mathcal{L}_{\sqrt{\eta_i},g}^{2m-2}(u)\right)\right) dv_g + 2\sum_{i=1}^{N} \int\limits_M \left(\left(\Delta_g^m u\right)\ \nabla\sqrt{\eta_i}, \nabla\left(\mathcal{P}_g^{(2m-1,1)}(u, \sqrt{\eta_i})\right)\right) dv_g$$

(2.36)

$$+ 2\sum_{i=1}^{N} \int\limits_M \left(\left(\Delta_g^m u\right)\ \nabla\sqrt{\eta_i}, \nabla\left(\mathcal{L}_{\sqrt{\eta_i},g}^{2m-2}(u)\right)\right) dv_g + 2\sum_{i=1}^{N} \int\limits_M \left(\nabla\left(\mathcal{P}_g^{(2m-1,1)}(u, \sqrt{\eta_i})\right), \nabla\left(\mathcal{L}_{\sqrt{\eta_i},g}^{2m-2}(u)\right)\right) dv_g$$

We have that

(2.37)

$$\sum_{i=1}^{N} \int\limits_M \left|\nabla\left(\mathcal{P}_g^{(2m-1,1)}(u, \sqrt{\eta_i})\right)\right|^2\ dv_g \le C\, \|u\|_{H^2_{2m}}^2 \quad \text{and} \quad \sum_{i=1}^{N} \int\limits_M \left|\nabla\left(\mathcal{L}_{\sqrt{\eta_i},g}^{2m-2}(u)\right)\right|^2\ dv_g \le C\, \|u\|_{H^2_{2m-1}}^2$$

while

$$\sum_{i=1}^{N} \int\limits_M \left(\sqrt{\eta_i}\ \nabla\left(\Delta_g^m u\right), \left(\Delta_g^m u\right)\ \nabla\sqrt{\eta_i}\right) dv_g = \sum_{i=1}^{N} \int\limits_M \left(\nabla\left(\Delta_g^m u\right), \left(\Delta_g^m u\right)\ \left(\sqrt{\eta_i}\ \nabla\sqrt{\eta_i}\right)\right) dv_g$$

(2.38)

$$= \frac{1}{2}\sum_{i=1}^{N} \int\limits_M \left(\nabla\left(\Delta_g^m u\right), \left(\Delta_g^m u\right)\ \nabla\eta_i\right) dv_g = \frac{1}{2}\int\limits_M \left(\nabla\left(\Delta_g^m u\right), \left(\Delta_g^m u\right)\ \nabla\left(\sum_{i=1}^{N}\eta_i\right)\right) dv_g = 0$$

And we obtain

$$
\begin{aligned}
&\left|\sum_{i=1}^{N}\int_M \left(\sqrt{\eta_i}\ \nabla\left(\Delta_g^m u\right), \nabla\left(\mathcal{P}_g^{(2m-1,1)}(u,\sqrt{\eta_i})\right)\right) dv_g\right| \\
=&\left|\sum_{i=1}^{N}\sum_{|l|=2m-1}\sum_{|\beta|=1}\int_M \left(\sqrt{\eta_i}\ \nabla\left(\Delta_g^m u\right), \nabla\left((a_{l,\beta}\partial_\beta\sqrt{\eta_i})\nabla^l u\right)\right) dv_g\right| \\
\leq&\left|\sum_{i=1}^{N}\sum_{|l|=2m}\sum_{|\beta|=1}\int_M \left(\sqrt{\eta_i}\ \nabla\left(\Delta_g^m u\right), (a_{l,\beta}\partial_\beta\sqrt{\eta_i})\nabla^l u\right) dv_g\right| \\
&+\left|\sum_{i=1}^{N}\sum_{|l|=2m-1}\sum_{|\beta|=1}\int_M \left(\nabla\left(\Delta_g^m u\right), \left(\sqrt{\eta_i}\ \nabla(a_{l,\beta}\partial_\beta\sqrt{\eta_i})\right)\nabla^l u\right) dv_g\right| \\
\leq&\left|\sum_{i=1}^{N}\sum_{|l|=2m}\sum_{|\beta|=1}\int_M \left(\sqrt{\eta_i}\ \nabla\left(\Delta_g^m u\right), (a_{l,\beta}\partial_\beta\sqrt{\eta_i})\nabla^l u\right) dv_g\right| \\
&+\sum_{i=1}^{N}\left|\sum_{|l|=2m-1}\sum_{|\beta|=1}\int_{B_{p_i}(\tau)} \left(\nabla\left(\Delta_g^m u\right), \left(\sqrt{\eta_i}\ \nabla(a_{l,\beta}\partial_\beta\sqrt{\eta_i})\right)\nabla^l u\right) dv_g\right|
\end{aligned}
\tag{2.39}
$$

Then we apply the integration by parts formula on each of the domains $\varphi^{-1}(B_{p_1}(\tau)) \subset \mathbb{R}^n$ to obtain

$$
\begin{aligned}
&\left|\sum_{i=1}^{N}\sum_{|l|=2m}\sum_{|\beta|=1}\int_M \left(\sqrt{\eta_i}\ \nabla\left(\Delta_g^m u\right), (a_{l,\beta}\partial_\beta\sqrt{\eta_i})\nabla^l u\right) dv_g\right| \\
&+\sum_{i=1}^{N}\left|\sum_{|l|=2m-1}\sum_{|\beta|=1}\int_{B_{p_i}(\tau)} \left(\nabla\left(\Delta_g^m u\right), \left(\sqrt{\eta_i}\ \nabla(a_{l,\beta}\partial_\beta\sqrt{\eta_i})\right)\nabla^l u\right) dv_g\right| \\
\leq&\left|\sum_{i=1}^{N}\sum_{|l|=2m}\sum_{|\beta|=1}\int_M \left(\sqrt{\eta_i}\ \nabla\left(\Delta_g^m u\right), (a_{l,\beta}\partial_\beta\sqrt{\eta_i})\nabla^l u\right) dv_g\right| + C\left\|u\right\|_{H_{2m}^2}^2 \\
\leq&\frac{1}{2}\left|\sum_{i=1}^{N}\sum_{|l|=2m}\sum_{|\beta|=1}\int_M \left(\nabla\left(\Delta_g^m u\right), (a_{l,\beta}\partial_\beta\eta_i)\nabla^l u\right) dv_g\right| + C\left\|u\right\|_{H_{2m}^2}^2 \\
\leq&\frac{1}{2}\left|\sum_{|l|=2m}\sum_{|\beta|=1}\int_M \left(\nabla\left(\Delta_g^m u\right), (a_{l,\beta}\partial_\beta(\sum_{i=1}^{N}\eta_i))\nabla^l u\right) dv_g\right| + C\left\|u\right\|_{H_{2m}^2}^2 \\
\leq& C\left\|u\right\|_{H_{2m}^2}^2 \qquad \text{since} \quad \sum_{i=1}^{N}\eta_i = 1
\end{aligned}
\tag{2.40}
$$

Similarly after integration by parts one obtains

$$\left|\sum_{i=1}^{N}\int_{M}(\sqrt{\eta_i}\ \nabla\left(\Delta_g^m u\right),\nabla\left(\mathcal{L}^{2m-2}_{\sqrt{\eta_i},g}(u)\right))\ dv_g\right|\leq C\left\|u\right\|^2_{H^2_{2m}} \tag{2.41}$$

$$\left|\sum_{i=1}^{N}\int_{M}((\Delta_g^m u)\ \nabla\sqrt{\eta_i},\nabla\left(\mathcal{P}_g^{(2m-1,1)}(u,\sqrt{\eta_i})\right))\ dv_g\right|\leq C\left\|u\right\|^2_{H^2_{2m}} \tag{2.42}$$

and

$$\begin{aligned}&\sum_{i=1}^{N}\int_{M}((\Delta_g^m u)\ \nabla\sqrt{\eta_i},\nabla\left(\mathcal{L}^{2m-2}_{\sqrt{\eta_i},g}(u)\right))\ dv_g\\ &+\sum_{i=1}^{N}\int_{M}(\nabla\left(\mathcal{P}_g^{(2m-1,1)}(u,\sqrt{\eta_i})\right),\nabla\left(\mathcal{L}^{2m-2}_{\sqrt{\eta_i},g}(u)\right))\ dv_g\leq C\left\|u\right\|^2_{H^2_{2m}}\end{aligned} \tag{2.43}$$

Hence for k odd, we also obtain that

$$\sum_{i=1}^{N}\int_{M}\left(\nabla\left(\Delta_g^m(\sqrt{\eta_i}u)\right)\right)^2 dv_g\leq\int_{M}\left(\nabla\left(\Delta_g^m u\right)\right)_g^2\ dv_g+C\left\|u\right\|^2_{H^2_{2m}} \tag{2.44}$$

Hence we have the claim and this completes the proof.

2.3. Best constant and direct Minimizaton

Let $\Omega_M\subset M$ be any smooth $n-$dimensional submanifold of M, possibly with boundary. In the sequel, we will either take $\Omega_M=M$, or $M\setminus\overline{B_{x_0}(\epsilon_0)}$ for some $\epsilon_0>0$ small enough. We define $H^2_{k,0}(\Omega_M)\subset H^2_k(M)$ as the completion of $C_c^\infty(\Omega_M)$ for the norm $\|\cdot\|_{H^2_k}$. In this section, we prove the following result in the spirit of Aubin [**3**]:

Theorem 2.3. *Let (M,g) be a compact Riemannian manifold of dimension $n>2k$, with $k\geq 1$. $\Omega_M\subset M$ be any smooth $n-$dimensional submanifold of M as above. Let P be a differential operator as in (2.3) and let $f\in C^{0,\theta}(\Omega_M)$ be a Hölder continuous positive function. Assume that P is coercive on $H^2_{k,0}(\Omega_M)$. Suppose that*

$$\inf_{u\in\mathcal{N}_f}\int_{\Omega_M}uP(u)\,dv_g<\frac{1}{\left(\sup_{\Omega_M}f\right)^{\frac{2}{2^\sharp_k}}K_0(n,k)}, \tag{2.45}$$

where

$$\mathcal{N}_f:=\{u\in H^2_{k,0}(\Omega_M):\int_{\Omega_M}f\left|u\right|^{2^\sharp_k}dv_g=1\} \tag{2.46}$$

Then there exists a minimizer $u\in\mathcal{N}_f$. Moreover, up to multiplication by a constant, $u\in C^{2k}(\overline{\Omega_M})$ is a solution to

$$\begin{cases}Pu=f\left|u\right|^{2^\sharp_k-2}u & \text{in } \Omega_M\\ D^\alpha u=0 & \text{on } \partial\Omega_M \quad \text{for } |\alpha|\leq k-1.\end{cases}$$

In addition, if the Green's function of P on Ω_M with Dirichlet boundary condition is positive, then any minimizer is either positive or negative. When $\Omega_M = M$, and the Green's function of P on M is positive, then up to changing sign, $u > 0$ is a solution to

$$Pu = fu^{2_k^\sharp - 1} \text{ in } M.$$

Proof of Theorem 2.3*:* This type of result is classical. We only sketch the proof. For simplicity, we take $\Omega_M = M$. The proof of the general case is similar. Here and in the sequel, we define (see (2.5))

$$I_P(u) := \int_M uP(u)\, dv_g \text{ for all } u \in H_k^2(M).$$

We start with the following lemma:

Lemma 2.3.1. *Let $(u_i) \in \mathcal{N}_f$ be a minimizing sequence for I_P on $\mathcal{N}_f$. Then*

(i) *Either there exists $u_0 \in \mathcal{N}_f$ such that $u_i \to u_0$ strongly in $H_k^2(M)$, and u_0 is a minimizer of I_P on $\mathcal{N}_f$*

(ii) *Or there exists $x_0 \in \overline{\Omega_M}$ such that $f(x_0) = \max_{\overline{\Omega_M}} f$ and $|u_i|^{2_k^\sharp}\, dv_g \rightharpoonup \delta_{x_0}$ as $i \to +\infty$ in the sense of measures. Moreover, $\inf\limits_{u \in \mathcal{N}_f} I_P(u) = \frac{1}{K_0(n,k)(\max_M f)^{\frac{2}{2_k^\sharp}}}$.*

Proof of Lemma 2.3.1*:* We define $\alpha := \inf\{I_P(u)/\, u \in \mathcal{N}_f\}$. As the functional I_g is coercive so the sequence (u_i) is bounded in $H_k^2(M)$. We let $u_0 \in H_k^2(M)$ such that, up to a subsequence, $u_i \rightharpoonup u_0$ weakly in $H_k^2(M)$ as $i \to +\infty$, and $u_i(x) \to u_0(x)$ as $i \to +\infty$ for a.e. $x \in M$. Therefore,

$$\|u_0\|_{L^{2_k^\sharp}}^{2_k^\sharp} \leq \liminf_{i \to +\infty} \|u_i\|_{L^{2_k^\sharp}}^{2_k^\sharp} = 1 \tag{2.47}$$

We define $v_i := u_i - u_0$. Up to extracting a subsequence, we have that $(v_i)_i \to 0$ in $H_{k-1}^2(M)$. We define $\mu_i := (\Delta_g^{k/2} u_i)^2\, dv_g$ and $\tilde{\nu}_i = |u_i|^{2_k^\sharp}\, dv_g$ and $\nu_i = f|u_i|^{2_k^\sharp}\, dv_g$ for all i. Up to a subsequence, we denote respectively by μ, $\tilde{\nu}$ and ν their limits in the sense of measures. It follows from the concentration-compactness Theorem 2.4 that,

$$\tilde{\nu} = |u_0|^{2_k^\sharp}\, dv_g + \sum_{j \in \mathcal{J}} \alpha_j \delta_{x_j} \text{ and } \mu \geq (\Delta_g^{k/2} u_0)^2\, dv_g + \sum_{j \in \mathcal{I}} \beta_j \delta_{x_i} \tag{2.48}$$

where $J \subset \mathbb{N}$ is at most countable, $(x_j)_{j \in J} \in M$ is a family of points, and $(\alpha_j)_{j \in J} \in \mathbb{R}_{\geq 0}$, $(\beta_j)_{j \in J} \in \mathbb{R}_{\geq 0}$ are such that

$$\alpha_j^{2/2_k^\sharp} \leq K_0(n,k)\, \beta_j \text{ for all } j \in J. \tag{2.49}$$

As a consequence, we get that

$$\nu = f|u_0|^{2_k^\sharp}\, dv_g + \sum_{j \in \mathcal{J}} f(x_j) \alpha_j \delta_{x_j} \tag{2.50}$$

Since $(u_i) \in \mathcal{N}_f$, and M is compact, we have that $\int_M d\nu = 1$ and then

$$1 = \int_M f|u_0|^{2_k^\sharp}\, dv_g + \sum_{j \in \mathcal{J}} f(x_j) \alpha_j. \tag{2.51}$$

Since $(u_i)_i \to u_0$ strongly in $H_{k-1}^2(M)$, integrating (2.48) yields

$$\alpha \geq I_P(u_0) + \sum_{j\in\mathcal{J}} \beta_j \geq \alpha \|u_0\|_{2_k^\sharp}^2 + K_0(n,k)^{-1} \sum_{j\in\mathcal{J}} \alpha_j^{2/2_k^\sharp}. \tag{2.52}$$

Since $\alpha \leq K_0(n,k)^{-1}(\max_M f)^{-2/2_k^\sharp}$, we then get that

(i) either $\|u_0\|_{2_k^\sharp} = 1$ and $\alpha_j = 0$ for all $j \in \mathcal{J}$,
(ii) or $u_0 \equiv 0$, $f(x_{j_0})\alpha_{j_0} = 1$ for some $j_0 \in \mathcal{J}$, $f(x_{j_0}) = \max_M f$ and $\alpha_j = 0$ for all $j \neq j_0$.

In case (i), we get from the strong convergence to 0 of $(v_i)_i$ in $H_{k-1}^2(M)$ that $I_P(u_i) = \int_M (\Delta_g^{k/2} v_i)^2\, dv_g + I_P(u_0) + o(1)$ as $i \to +\infty$. Since $u_0 \in \mathcal{N}_f$ and (u_i) is a minimizing sequence, we then get that $(v_i)_0$ goes to 0 strongly in $H_k^2(M)$, and therefore $u_i \to u_0$ strongly in $H_k^2(M)$.

In case (ii), (2.52) yields $\alpha = K_0(n,k)^{-1}(\max_M f)^{-2/2_k^\sharp}$ and $I_P(u_0) = 0$, which yields $u_0 \equiv 0$ since the operator is coercive.

This completes the proof of Lemma 2.3.1. □

We go back to the proof of Theorem 2.3. Let $(u_i)_i$ be a minimizing sequence for I_P on $\mathcal{N}_f$. It follows from the assumption (2.45) that case (i) of Lemma 2.3.1 holds, and then, there exists a minimizer $u_0 \in \mathcal{N}_f$ that is a minimizer. Therefore, it is a weak solution to $P_g^k u_0 = \alpha f\, |u_0|^{2_k^\sharp - 2}\, u_0$ in M (see (2.145) for the definition). It then follows from the regularity Theorem 2.8.3 that $u \in C^{2k,\theta}(M)$.

We let $G : M \times M \setminus \{(x,x)/\, x \in M\}$ be the Green's function of P on M. We assume that $G(x,y) > 0$ for all $x \neq y \in M$. Green's representation formula yields

$$\varphi(x) = \int_M G(x,y)(P\varphi)(y)\, dv_g \text{ for all } x \in M \text{ and all } \varphi \in C^{2k}(M). \tag{2.53}$$

It follows from Proposition 2.8.2 that there exists $v \in H_k^2(M)$ such that

$$Pv = \alpha f\, |u_0|^{2_k^\sharp - 1} \qquad \text{in } M. \tag{2.54}$$

Standard regularity (taking inspiration from Vand der Vorst [**35**]) yields $v \in C^{2k}(M)$. We have that $P\,(v \pm u_0) \geq 0$. Since $G > 0$, it follows from Green's formula (2.53) that $v \pm u_0 \geq 0$. So $v \geq |u_0|$ and therefore $v \neq 0$. Independently, since $Pv \geq 0$ and $v \not\equiv 0$, Green's formula (2.53) yields $v > 0$. Using Hölder's inequality and $v \geq |u_0|$, we get that

$$J_P(u) = \frac{\int_M vP(v)\ dv_g}{\left(\int_M f|v|^{2^\sharp_k}\ dv_g\right)^{2/2^\sharp_k}} = \frac{\alpha \int_M vf\,|u_0|^{2^\sharp_k - 1}\ dv_g}{\left(\int_M f|v|^{2^\sharp_k}\ dv_g\right)^{2/2^\sharp_k}} \tag{2.55}$$

$$\leq \frac{\alpha \left(\int_M f|v|^{2^\sharp_k}\ dv_g\right)^{\frac{1}{2^\sharp_k}} \left(\int_M f|u_0|^{2^\sharp_k}\ dv_g\right)^{\frac{2^\sharp_k - 1}{2^\sharp_k}}}{\left(\int_M f|v|^{2^\sharp_k}\ dv_g\right)^{2/2^\sharp_k}} \tag{2.56}$$

$$\leq \frac{\alpha \left(\int_M f|u_0|^{2^\sharp_k}\ dv_g\right)^{\frac{2^\sharp_k - 1}{2^\sharp_k}}}{\left(\int_M f|u_0|^{2^\sharp_k}\ dv_g\right)^{1/2^\sharp_k}} \leq \alpha \left(\int_M f|u_0|^{2^\sharp_k}\ dv_g\right)^{\frac{2^\sharp_k - 2}{2^\sharp_k}} \leq \alpha \tag{2.57}$$

since $\int_M f\,|u_0|^{2^\sharp_k}\,dv_g = 1$. Since α is the infimum of the functional, we get that $J_P(u) = \alpha$. Hence v attains the infimum and therefore it also solves the equation $Pv = \mu f v^{2^\sharp_k - 1}$ weakly in M, and $v \in H^2_{k,0}(M)$. Moreover, one has equality in all the inequalities above, and then $|u_0| = v > 0$, and therefore either $u_0 > 0$ or $u_0 < 0$ in M. This ends the proof of Theorem 2.3. $\square$

2.4. Concentration Compactness Lemma

We now state and prove the concentration compactness lemma in the spirit of P.-L.Lions for the case of a closed manifold:

Theorem 2.4 (Concentration-compactness)**.** *Let (M, g) be a smooth, compact Riemannian manifold of dimension n and let k be a positive integer such that $2k < n$. Suppose (u_m) be a bounded sequence in $H^2_k(M)$. Up to extracting a subsequence, there exist two nonnegative Borel-regular measure μ, ν on M and $u \in H^2_k(M)$ such that*

(a) $u_m \rightharpoonup u$ *weakly in $H^2_k(M)$*

(b) $\mu_m := (\Delta_g^{k/2} u_m)^2\ dv_g \rightharpoonup \mu$ *weakly in the sense of measures*

(c) $\nu_m := |u_m|^{2^\sharp_k}\ dv_g \rightharpoonup \nu$ *weakly in the sense of measures*

Then there exists an at most countable index set $\mathcal{I}$, a family of distinct points $\{x_i \in M : i \in \mathcal{I}\}$, families of nonnegative weights $\{\alpha_i : i \in \mathcal{I}\}$ and $\{\beta_i : i \in \mathcal{I}\}$ such that

(i)

$$\nu = |u|^{2_k^\sharp}\, dv_g + \sum_{i\in\mathcal{I}} \alpha_i \delta_{x_i} \tag{2.58}$$

$$\mu \geq (\Delta_g^{k/2} u)^2\, dv_g + \sum_{i\in\mathcal{I}} \beta_i \delta_{x_i} \tag{2.59}$$

where δ_x denotes the Dirac mass at $x \in M$ with mass equal to 1.

(ii) *for all $i \in \mathcal{I}$, $\alpha_i^{2/2_k^\sharp} \leq K_0(n,k)\ \beta_i$. In particular $\sum\limits_{i\in\mathcal{I}} \alpha_i^{2/2_k^\sharp} < \infty$.*

Proof of Theorem 2.4*:* By the Riesz representation theorem (μ_m), and (ν_m) are sequences of Radon measures on M.

Step 1: First we assume that $u \equiv 0$. Let $\varphi \in C^\infty(M)$, then from (2.2) we have that, given any $\varepsilon > 0$ there exists $B_\varepsilon \in \mathbb{R}$ such that

(2.60)
$$\left(\int\limits_M |\varphi u_m|^{2_k^\sharp}\ dv_g\right)^{2/2_k^\sharp} \leq (K_0(n,k)+\varepsilon)\int\limits_M (\Delta_g^{k/2}(\varphi u_m))^2\ dv_g + B_\varepsilon ||\varphi u_m||^2_{H^2_{k-1}}$$

Since $u_m \rightharpoonup 0$ in $H_k^2(M)$, letting $m \to +\infty$ and then taking the limit $\varepsilon \to 0$, it follows that

$$\left(\int\limits_M |\varphi|^{2_k^\sharp}\ d\nu\right)^{2/2_k^\sharp} \leq K_0(n,k)\int\limits_M \varphi^2\ d\mu \tag{2.61}$$

By regularity of the Borel measure ν, (2.61) holds for any Borel measurable function φ and in particular for any Borel set $E \subset M$ we have

$$\nu(E)^{2/2_k^\sharp} \leq K_0(n,k)\ \mu(E) \tag{2.62}$$

Therefore the measure ν is absolutely continuous with respect to the measure μ and hence by the Radon-Nikodyn theorem, we get

$$d\nu = f d\mu \text{ and } d\mu = g d\nu + d\sigma \tag{2.63}$$

where $f \in L^1(M,\mu)$ and $g \in L^1(M,\nu)$ are nonnegative functions, σ is a positive Borel measure on M and $d\nu \perp d\sigma$.

Let $S = M\backslash(supp\ \sigma)$. Then for any $\varphi \in C(M)$ with support $supp(\varphi) \subset S$ one has

$$\int\limits_M \varphi\ d\nu = \int\limits_M \varphi f\ d\mu = \int\limits_M \varphi\ fg\ d\nu \tag{2.64}$$

By regularity of the Borel measures μ and ν (2.64) holds for any Borel measurable function φ. This implies that $fg = 1$ a.e with respect to ν. So, in particular $g > 0$

ν a.e in S. Let $\psi \in C(M)$, taking $\varphi = \psi\chi_S$ in (2.61) we have

$$\left(\int\limits_M |\psi|^{2_k^\sharp} \mathcal{X}_S \, d\nu\right)^{2/2_k^\sharp} \leq K_0(n,k) \int\limits_M \psi^2 \mathcal{X}_S \, d\mu$$

$$= K_0(n,k) \int\limits_M \psi^2 \mathcal{X}_S \, [g d\nu + d\sigma] = K_0(n,k) \int\limits_M \psi^2 g \mathcal{X}_S \, d\nu \tag{2.65}$$

Since $d\nu \perp d\sigma$ and *supp* $\nu \subset S$, we get that

$$\left(\int\limits_M |\psi|^{2_k^\sharp} \, d\nu\right)^{2/2_k^\sharp} \leq K_0(n,k) \int\limits_M \psi^2 g \, d\nu \tag{2.66}$$

By regularity of the Borel measure ν the above relation holds for any Borel measurable function ψ.

Let $\phi \in C(M)$ and let $\psi = \phi g^{\frac{1}{2_k^\sharp - 2}} \mathcal{X}_{\{g \leq N\}}$, $d\nu_N = g^{\frac{2_k^\sharp}{2_k^\sharp - 2}} \mathcal{X}_{\{g \leq N\}} d\nu$. Then we have

$$\left(\int\limits_M |\phi|^{2_k^\sharp} \, d\nu_N\right)^{2/2_k^\sharp} \leq K_0(n,k) \int\limits_M \phi^2 \, d\nu_N \tag{2.67}$$

By regularity of the Borel measure ν the above relation holds for any Borel measurable function ϕ.
It follows from Proposition 2.4.1 below that for each N there exist a finite set $\mathcal{I}_N$, a finite set of distinct points $\{x_i : i \in \mathcal{I}_N\}$and a finite set of weights $\{\tilde{\alpha}_i : i \in \mathcal{I}_N\}$ such that

$$d\nu_N = \sum_{i \in \mathcal{I}_N} \tilde{\alpha}_i \, \delta_{x_i} \tag{2.68}$$

Let $\mathcal{I} = \bigcup\limits_{N=1}^{\infty} \mathcal{I}_N$. Then $\mathcal{I}$ is a countable set. For a Borel set E, then one has by monotone convergence theorem

$$\int\limits_M \chi_E \, g^{\frac{2_k^\sharp}{2_k^\sharp - 2}} d\nu = \lim_{N \to \infty} \int\limits_M \chi_E \, d\nu_N \tag{2.69}$$

So $g^{\frac{2_k^\sharp}{2_k^\sharp - 2}} d\nu = \sum\limits_{i \in \mathcal{I}} \tilde{\alpha}_i \delta_{x_i}$. Since $g > 0$ ν a.e , there exists $\alpha_i > 0$ such that we have $d\nu = \sum\limits_{i \in \mathcal{I}} \alpha_i \delta_{x_i}$. Since $\mu = g d\nu + d\sigma \geq g d\nu$, we get that

$$\mu \geq \sum_{i \in \mathcal{I}} \beta_i \delta_{x_i} \qquad \text{where } \beta_i = g(x_i)\alpha_i \tag{2.70}$$

Taking $\psi = \mathcal{X}_{\{x_i\}}$ in (2.66) we have for all $i \in \mathcal{I}$

$$\alpha_i^{2/2_k^\sharp} \leq K_0(n,k) \, g\alpha_i = K_0(n,k) \, \beta_i \tag{2.71}$$

and

$$\frac{1}{K_0(n,k)}\sum_{i\in\mathcal{I}}\alpha_i^{2/2_k^\sharp}\leq\sum_{i\in\mathcal{I}}\beta_i\leq\mu(M)<+\infty \tag{2.72}$$

This proves the theorem for $u\equiv 0$. This ends Step 1.

Step 2: Assume $u\not\equiv 0$ and let $v_m := u_m - u$. Then $v_m \rightharpoonup 0$ weakly in $H_k^2(M)$. Therefore, as one checks, $\tilde{\mu}_m := (\Delta_g^{k/2}v_m)^2\,dv_g \rightharpoonup \mu - (\Delta_g^{k/2}u)^2\,dv_g$ and $\tilde{\nu}_m := |v_m|^{2_k^\sharp}\,dv_g \rightharpoonup \nu - |u|^{2_k^\sharp}\,dv_g$ weakly in the sense of measures. Applying Step 1 to the measures $\tilde{\mu}_m$ and $\tilde{\nu}_m$ yields Theorem 2.4. □

We now prove the reversed Hölder inequality that was used in the proof.

Proposition 2.4.1. *Let μ be a finite Borel measure on M and suppose that for any Borel measurable function φ one has*

$$\left(\int_M|\varphi|^q\,d\mu\right)^{1/q}\leq C\left(\int_M|\varphi|^p\,d\mu\right)^{1/p} \tag{2.73}$$

for some $C>0$ and $1\leq p<q<+\infty$. Then there exists j points $x_1,\dots,x_j\in M$, and j positive real numbers $c_1,\dots,c_j$ such that

$$\mu=\sum_{i=1}^{j}c_i\delta_{x_i} \tag{2.74}$$

where δ_x denotes the Dirac measure concentrated at $x\in M$ with mass equal to 1. Moreover $c_i\geq(\frac{1}{C})^{\frac{pq}{q-p}}$.

Proof. Let E be a Borel set in M. Taking $\varphi=\chi_E$ we obtain that, either $\mu(E)=0$ or $\mu(E)\geq(\frac{1}{C})^{\frac{pq}{q-p}}$

We define $\mathcal{O}:=\{x\in M : \text{for some } r>0\ \ \mu(B_x(r))=0\}$. Then $\mathcal{O}$ is open. Now if $K\subset\mathcal{O}$ is compact, then K can be covered by a finite number of balls each of which has measure 0, therefore $\mu(K)=0$. By the regularity of the measure hence it follows that $\mu(\mathcal{O})=0$. If $x\in M\backslash\mathcal{O}$, then for all $r>0$ one has $\mu(B_x(r))\geq(\frac{1}{C})^{\frac{pq}{q-p}}$. Then

$$\mu(\{x\})=\lim_{m\to+\infty}\mu(B_x(1/m))\geq\left(\frac{1}{C}\right)^{\frac{pq}{q-p}} \tag{2.75}$$

Since the measure μ is finite, this implies that that the set $M\backslash\mathcal{O}$ is finite. So let $M\backslash\mathcal{O}=\{x_1,\cdots,x_j\}$, therefore for any borel set E in M

$$\mu(E)=\mu(E\cap\{x_1,\cdots,x_j\})=\sum_{x_i\in E}\mu(\{x_i\})=\sum_{i=1}^{j}\mu(\{x_i\})\delta_{x_i}(E) \tag{2.76}$$

Hence the lemma follows with $c_i=\mu(\{x_i\})$. □

2.5. Topological method of Coron

In this section we obtain higher energy solutions by Coron's topological method if the functional J_P does not have a minimizer, for the case $f \equiv 1$. This will complete the proof of the first part of Theorem 2.1, that is the existence of solutions to (2.4) with no sign-restriction. For $\mu > 0$ and $y_0 \in \mathbb{R}^n$, we define

$$\mathcal{B}_{y_0,\mu}(y) = \alpha_{n,k} \left(\frac{\mu}{\mu^2 + |y - y_0|^2} \right)^{\frac{n-2k}{2}} \tag{2.77}$$

where the choice of $\alpha_{n,k}$'s are such that for all μ, $\|\mathcal{B}_{y_0,\mu}\|_{L^{2^\sharp_k}} = 1$ and $\|\mathcal{B}_{y_0,\mu}\|^2_{\mathscr{D}^{k,2}} = \frac{1}{K_0(n,k)}$. These functions are the extremal functions of the Euclidean Sobolev Inequality (2.10) and they satisfy the equation

$$\Delta^k \mathcal{B}_{y_0,\mu} = \frac{1}{K_0(n,k)} \mathcal{B}_{y_0,\mu}^{2^\sharp_k - 1} \qquad \text{in } \mathbb{R}^n \tag{2.78}$$

Let $\tilde{\eta}_r \in C_c^\infty(\mathbb{R}^n)$, $0 \le \tilde{\eta}_r \le 1$ be a smooth cut-off function, such that $\tilde{\eta}_r = 1$ for $x \in B_0(r)$ and $\tilde{\eta}_r = 1$ for $x \in \mathbb{R}^n \backslash B_0(2r)$. Let $\iota_g > 0$ be the injectivity radius of (M, g). For any $p \in M$, we let η_p be a smooth cut-off function on M such that

$$\eta_p(x) = \begin{cases} \tilde{\eta}_{\frac{\iota_g}{10}}(exp_p^{-1}(x)) & \text{for } x \in B_p(\iota_g) \subset M \\ 0 & \text{for } x \in M \backslash B_p(\iota_g) \end{cases} \tag{2.79}$$

For any $x \in M$, we define

$$\mathcal{B}^M_{p,\mu}(x) = \eta_p(x) \; \mathcal{B}_{0,\mu}(exp_p^{-1}(x)) \tag{2.80}$$

$\mathcal{B}^M_{p,\mu}$ is the standard bubble centered at the point $p \in M$ and with radius μ

$$\mathcal{B}^M_{p,\mu}(x) = \alpha_{n,k} \eta_p(x) \left(\frac{\mu}{\mu^2 + d_g(p,x)^2} \right)^{\frac{n-2k}{2}} \tag{2.81}$$

We have

Proposition 2.5.1. *Let (M, g) be a smooth, compact Riemannian manifold of dimension n and let k be a positve integer such that $2k < n$. Consider the functional J_P on the space $H_k^2(M) \backslash \{0\}$. Then the sequence of functions $\left(\mathcal{B}^M_{p,\mu}\right) \in C^\infty(M)$ defined above is such that:*

$$\begin{array}{lll} (a) & \lim\limits_{\mu \to 0} J_P(\mathcal{B}^M_{p,\mu}) = \frac{1}{K_0(n,k)} & \textit{uniformly for } p \in M \\ (b) & \lim\limits_{\mu \to 0} \left\|\mathcal{B}^M_{p,\mu}\right\|_{L^{2^\sharp_k}} = 1 & \textit{uniformly for } p \in M \\ (c) & \mathcal{B}^M_{p,\mu} \rightharpoonup 0 & \textit{weakly in } H_k^2(M), \textit{ as } \mu \to 0 \end{array}$$

Proof of Proposition 2.5.1*:* We claim that (c) holds. We first prove that $\mathcal{B}_{p,\mu}^M$ is uniformly bounded in $H_k^2(M)$. Indeed,

$$\begin{aligned}
\sum_{\alpha\le k}\int_M \left(\Delta_g^{\alpha/2}\,\mathcal{B}_{p,\mu}^M\right)^2\,dv_g &\le \sum_{\alpha\le k}\int_{B_p(\iota_g/5)} \left(\Delta_g^{\alpha/2}\,\mathcal{B}_{p,\mu}^M\right)^2\,dv_g\\
&\le C\sum_{l\le k}\int_{B_0(\iota_g/5)} \left|\nabla^l\,\mathcal{B}_{p,\mu}^M\circ \exp_p\right|^2\,dx\\
&\le \sum_{l\le k}\int_{B_0(\iota_g/5)} \left|\nabla^l\left(\frac{\mu}{\mu^2+|x|^2}\right)^{\frac{n-2k}{2}}\right|^2\,dx\\
&\le \sum_{l\le k}\int_{B_0(\iota_g/(5\mu))} \mu^{2(k-l)}\left|\nabla^l\left(1+|x|^2\right)^{-\frac{n-2k}{2}}\right|^2\,dx.
\end{aligned}$$

As one checks, the right-hand-side is uniformly bounded wrt $\mu\to 0$, so $(\mathcal{B}_{p,\mu}^M)$ is uniformly bounded wrt p and $\mu\to 0$. Moreover, the above computations yield $\int_M(\mathcal{B}_{p,\mu}^M)^2\,dv_g\to 0$ as $\mu\to 0$. Therefore, $\mathcal{B}_{p,\mu}^M\rightharpoonup 0$ as $\mu\to 0$ uniformly wrt $p\in M$. This proves the claim.

The space $H_k^2(M)$ is compactly embedded in $H_{k-1}^2(M)$. Therefore $\mathcal{B}_{p,\mu}^M\to 0$ in $H_{k-1}^2(M)$ as $\mu\to 0$. Hence

$$\lim_{\mu\to 0}\left(\sum_{l=0}^{k-1}\int_M A_l(g)(\nabla^l\mathcal{B}_{p,\mu}^M,\nabla^l\mathcal{B}_{p,\mu}^M)\,dv_g\right)=0 \tag{2.82}$$

Now we estimate the term $\int_M |\,\mathcal{B}_{p,\mu}^M|^{2_k^\sharp}\,dv_g$. We fix $R>0$. We claim that

$$\lim_{R\to+\infty}\lim_{\mu\to 0}\int_{M\setminus B_p(\mu R)} |\,\mathcal{B}_{p,\mu}^M|^{2_k^\sharp}\,dv_g=0 \tag{2.83}$$

Now for μ sufficiently small

$$\begin{aligned}
\int_{M\setminus B_p(\mu R)} |\,\mathcal{B}_{p,\mu}^M|^{2_k^\sharp}\,dv_g &= \int_{B_p(\iota_g)\setminus B_p(\mu R)} |\,\mathcal{B}_{p,\mu}^M|^{2_k^\sharp}\,dv_g\\
&= \int_{B_0(\iota_g)\setminus B_0(\mu R)} |\,\mathcal{B}_{p,\mu}^M(exp_p(y))|^{2_k^\sharp}\,\sqrt{|g(exp_p(y))|}\,dy\\
&\le \int_{B_0(\frac{\iota_g}{\mu})\setminus B_0(R)} |\mathcal{B}_{0,1}(y)|^{2_k^\sharp}\,\sqrt{|g(exp_p(\mu y))|}\,dy. \qquad (2.84)
\end{aligned}$$

Since $\mathcal{B}_{0,1}\in L^{2_k^\sharp}(\mathbb{R}^n)$, this yields the claim.

Similarly, for μ sufficiently small

$$\int\limits_{B_p(\mu R)} |\ \mathcal{B}^M_{p,\mu}|^{2^\sharp_k}\ dv_g = \int\limits_{B_0(\mu R)} |\ \mathcal{B}^M_{p,\mu}(exp_p(y))|^{2^\sharp_k}\ \sqrt{|g(exp_p(y))|}\ dy \tag{2.85}$$

$$= \int\limits_{B_0(R)} |\ \mathcal{B}_{0,1}|^{2^\sharp_k}\ \sqrt{|g(exp_p(\mu y))|}\ dy \tag{2.86}$$

$$= \int\limits_{B_0(R)} |\ \mathcal{B}_{0,1}|^{2^\sharp_k}\ dy + o\left(\|\mathcal{B}_{0,1}\|_{L^{2^\sharp_k}}\right) \qquad \text{as } \mu \to 0 \tag{2.87}$$

Therefore

$$\lim_{\mu\to 0}\int\limits_M |\ \mathcal{B}^M_{p,\mu}|^{2^\sharp_k}\ dv_g = \int\limits_{\mathbb{R}^n} |\ \mathcal{B}_{0,1}|^{2^\sharp_k}\ dv_g \tag{2.88}$$

So we have (b).

Finally we estimate the term $\int\limits_M (\Delta_g^{k/2}\ \mathcal{B}^M_{p,\mu})^2\ dv_g$. We fix $R > 0$. By calculating in terms of the local coordinates given by exp_p, we get for μ sufficiently small

$$\int\limits_{B_p(\mu R)} (\Delta_g^{k/2}\ \mathcal{B}^M_{p,\mu})^2\ dv_g = \int\limits_{B_0(R)} (\Delta^{k/2}\mathcal{B}_{0,1})^2\ dy + o\,(1) \qquad \text{as } \mu \to 0. \tag{2.89}$$

We claim that

$$\lim_{R\to+\infty}\lim_{\mu\to 0} \int\limits_{M\setminus B_p(\mu R)} (\Delta_g^{k/2}\ \mathcal{B}^M_{p,\mu})^2\ dv_g = 0. \tag{2.90}$$

We prove the claim. Indeed, via the exponential map at p, we have that

$$\int\limits_{M\setminus B_p(\mu R)} (\Delta_g^{k/2}\ \mathcal{B}^M_{p,\mu})^2\ dv_g = \int\limits_{B_p(\iota_g)\setminus B_p(\mu R)} (\Delta_g^{k/2}\ \mathcal{B}^M_{p,\mu})^2\ dv_g \tag{2.91}$$

$$= \int\limits_{B_0(\iota_g)\setminus B_0(\mu R)} (\Delta^{k/2}_{exp_p^* g}\ \mathcal{B}_{0,\mu})^2\ dv_{exp_p^* g} \tag{2.92}$$

$$\le C\sum_{|\alpha|=0}^{k}\ \int\limits_{B_0(\iota_g)\setminus B_0(\mu R)} |D^\alpha(\tilde\eta_{\frac{\iota_g}{10}}\mathcal{B}_{0,\mu})|^2\ dx \tag{2.93}$$

Since $\mathcal{B}_{0,\mu} \to 0$ strongly in $H^2_{k-1,loc}(\mathbb{R}^n)$, then, as $\mu \to 0$, we have that

$$\int\limits_{M\setminus B_p(\mu R)} (\Delta_g^{k/2}\ \mathcal{B}^M_{p,\mu})^2\ dv_g \le C \int\limits_{B_0(\iota_g)\setminus B_0(\mu R)} \tilde\eta^2_{\frac{\iota_g}{10}}|D^k\mathcal{B}_{0,\mu}|^2\ dx + o(1) \tag{2.94}$$

$$\le C \int\limits_{B_0(\iota_g/\mu)\setminus B_0(R)} |D^k\mathcal{B}_{0,1}|^2\ dx + o(1) \le C \int\limits_{\mathbb{R}^n\setminus B_0(R)} |D^k\mathcal{B}_{0,1}|^2\ dx + o(1). \tag{2.95}$$

Since $D^k\mathcal{B}_{0,1} \in L^2(\mathbb{R}^n)$, this yields (2.90). This proves the claim.

Equations (2.89) and (2.90) yield (a) and (b) of Proposition 2.5.1 for any fixed $p \in M$. Since the manifold M is compact, we note that in the above calculations there is no dependence on the point p of the closed manifold M. So the convergence is uniform for all points $p \in M$. This ends the proof of Proposition 2.5.1. □

Fix some θ such that $\frac{1}{K_0(n,k)} + 4\theta < 2^{2k/n}\frac{1}{K_0(n,k)}$. Then from (2.5.1) it follows that, there exists μ_0 small, such that for all $\mu \in (0, \iota_g \mu_0)$ and for all $p \in M$ we have

$$J_P(\mathcal{B}^M_{p,\mu}) \leq \frac{1}{K_0(n,k)} + \theta \tag{2.96}$$

We fix $x_0 \in M$, and we assimilate isometrically $T_{x_0}M$ to $\mathbb{R}^n$, and we define the sphere $S^n := \{x \in \mathbb{R}^n / \|x\| = 1\}$. For $(\sigma, t) \in S^n \times [0, \iota_g/2)$, we define $\sigma^M_t := exp_{x_0}(t\sigma)$ and

$$u^\sigma_t(x) = \alpha_{n,k}\eta_{\sigma^M_t}(x)\left[\frac{\mu_0(\iota_g/2 - t)}{(\mu_0(\iota_g/2-t))^2 + d_g(\sigma^M_t, x)^2}\right]^{\frac{n-2k}{2}} = \mathcal{B}^M_{\sigma^M_t, \mu_0(\iota_g/2-t)} \tag{2.97}$$

It then follows from our previous step and the choice of μ_0 in (2.96)

$$J_P(u^\sigma_t) \leq \frac{1}{K_0(n,k)} + \theta \qquad \forall (\sigma, t) \in S^n \times [0, \iota_g/2). \tag{2.98}$$

Let $\eta \in C^\infty_c(\mathbb{R}^n)$ be a smooth, nonnegative, cut-off function such that $\eta(x) = 1$ for $|x| \geq 1/2$ and $\eta(x) = 0$ for $|x| < 1/4$. For $R \geq 1$, let η_R be a smooth, nonnegative, cut-off function, such that

$$\eta_R(x) = \begin{cases} 1 & \text{if} \quad d_g(x_0, x) \geq \frac{\iota_g}{10R} \\ \eta\left(\frac{10R}{\iota_g} exp^{-1}_{x_0}(x)\right) & \text{if} \quad d_g(x_0, x) < \frac{\iota_g}{10R} \end{cases} \tag{2.99}$$

Then the functions η_R are such that $\eta_R(x) = 1$ if $d_g(x_0, x) \geq \frac{\iota_g}{20R}$ and $\eta_R(x) = 0$ if $d_g(x_0, x) < \frac{\iota_g}{40R}$. We define

$$v^\sigma_{t,R}(x) := \eta_R(x)\ u^\sigma_t(x) \text{ for all } x \in M. \tag{2.100}$$

Then we have

Proposition 2.5.2.

$$\lim_{R\to+\infty} v^\sigma_{t,R} = u^\sigma_t \quad \textit{in } H^2_k(M) \quad \textit{uniformly } \forall (\sigma, t) \in S^n \times [0, \iota_g/2). \tag{2.101}$$

Proof of Proposition 2.5.2*:* We first note that for all $(\sigma, t) \in S^n \times [0, \iota_g/2)$ the functions u^σ_t are uniformly bounded in C^{2k}-norm in the ball $B_{x_0}(\frac{\iota_g}{20}) \subset M$. And for any nonnegative integer α, one has $|\nabla^\alpha_g \eta_R|_g \leq CR^\alpha$. Therefore

$$\left\|v^\sigma_{t,R} - u^\sigma_t\right\|^2_{H^2_k} = \sum_{\alpha=0}^k \int_M (\Delta_g^{\alpha/2}(v^\sigma_{t,R} - u^\sigma_t))^2\ dv_g \tag{2.102}$$

$$= \sum_{\alpha=0}^k \int_{B_{x_0}(\frac{\iota_g}{20R})} (\Delta_g^{\alpha/2}(v^\sigma_{t,R} - u^\sigma_t))^2\ dv_g \tag{2.103}$$

$$= \sum_{\alpha=0}^k \int_{B_{x_0}(\frac{\iota_g}{20R})} (\Delta_g^{\alpha/2}((\eta_R - 1)u^\sigma_t))^2\ dv_g = O\left(\frac{1}{R}\right) \qquad (\text{ as } n \geq 2k+1\ ) \tag{2.104}$$

The above convergence is uniform $w.r.t$ $(\sigma,t)\in S^n\times[0,\iota_g/2)$. This proves Proposition 2.5.2. $\square$

So it follows that, there exists $R_0>0$, large, such that for all $R\geq R_0$ one has

$$J_P(v^\sigma_{t,R})\leq J_P(u^\sigma_t)+2\theta<2^{2k/n}\frac{1}{K_0(n,k)}\qquad\forall\ (\sigma,t)\in S^n\times[0,\iota_g/2)\tag{2.105}$$

As one checks for any $(\sigma,t)\in S^n\times[0,\iota_g/2)$, the functions $v^\sigma_{t,R_0}\neq 0$, and has support in $M\backslash B_{x_0}(\iota_g/40R_0)$. Let $\epsilon_0>0$ be such that $M\backslash B_{x_0}(\iota_g/40R_0)\subset M\backslash B_{x_0}(\epsilon_0)$ and we define

$$\Omega_{\epsilon_0}:=M\backslash\overline{B_{x_0}(\epsilon_0)}\tag{2.106}$$

Then for any $(\sigma,t)\in S^n\times[0,\iota_g/2)$ the functions $v^\sigma_{t,R_0}\in H^2_{k,0}(\Omega_{\epsilon_0})\backslash\{0\}$. Propositions 2.5.1 and 2.5.2 yield

$$\lim_{t\to\iota_g/2}J_P(v^\sigma_{t,R_0})=\frac{1}{K_0(n,k)}\qquad\text{uniformly for all }\sigma\in S^n.\tag{2.107}$$

Also v^σ_{0,R_0} is a fixed function independent of σ and

$$v^\sigma_{t,R_0}\rightharpoonup\delta_{exp_{x_0}(\frac{\iota_g}{2}\sigma)}\qquad\text{weakly in the sense of measures as }t\to\iota_g/2\tag{2.108}$$

We define $S_k:=K_0(n,k)^{-1}$. For any $c\in\mathbb{R}$, we define the sublevel sets of the functional I_P on $\mathcal{N}_{\epsilon_0}$

$$\mathcal{I}_c:=\{u\in\mathcal{N}_{\epsilon_0}:I_P(u)<c\}\tag{2.109}$$

where $\mathcal{N}_{\epsilon_0}:=\{u\in H^2_{k,0}(\Omega_{\epsilon_0})/\ \|u\|_{2^\sharp_k}=1\}$.

Proposition 2.5.3. *Suppose $I_P(u)>\frac{1}{K_0(n,k)}$ for all $u\in\mathcal{N}_{\epsilon_0}$, then there exists $\sigma_0>0$ for which there exists a continuous map*

$$\Gamma:\mathcal{I}_{S_k+\sigma_0}\longrightarrow\overline{\Omega}_{\epsilon_0}\tag{2.110}$$

such that if $(u_i)\in\mathcal{I}_{S_k+\sigma_0}$ is a sequence such that $|u_i|^{2^\sharp_k}\,dv_g\rightharpoonup\delta_{p_0}$ weakly in the sense of measures, for some point $p_0\in\overline{\Omega}_{\epsilon_0}$, then

$$\lim_{i\to+\infty}\Gamma(u_i)=p_0\tag{2.111}$$

Proof of Proposition 2.5.3*:* By the Whitney embedding theorem, the manifold M admits a smooth embedding into $\mathbb{R}^{2n+1}$. If we denote this embedding by $\mathcal{F}:M\to\mathbb{R}^{2n+1}$, then M is diffeomorphic to $\mathcal{F}(M)$ where $\mathcal{F}(M)$ is an embedded submanifold of $\mathbb{R}^{2n+1}$. For $u\in\mathcal{N}_{\epsilon_0}$, we define

$$\tilde\Gamma(u):=\int_{\Omega_M}\mathcal{F}(x)\ |u(x)|^{2^\sharp_k}\ dv_g(x)\tag{2.112}$$

Then $\tilde\Gamma:\mathcal{N}_{\epsilon_0}\to\mathbb{R}^{2n+1}$ is continuous. Next we claim that for every $\epsilon>0$ there exists a $\sigma>0$ such that

$$u\in\mathcal{I}_{S_k+\sigma}\ \Rightarrow\ dist\left(\tilde\Gamma(u),\mathcal{F}(\overline{\Omega}_{\epsilon_0})\right)<\epsilon\tag{2.113}$$

Suppose that the claim is not true, then there exists an $\epsilon' > 0$ and a sequence $(u_i) \in \mathcal{N}_{\epsilon_0}$, such that $\lim_{i\to+\infty} I_P(u_i) = S_k$ and $dist\left(\tilde\Gamma(u), \mathcal{F}(\overline{\Omega}_{\epsilon_0})\right) \geq \epsilon'$. Since there is no minimizer for I_P on $\mathcal{N}_{\epsilon_0}$, it follows from Lemma 2.3.1 that for such a sequence (u_i) there exists a point $p_0 \in \overline{\Omega}_{\epsilon_0}$ such that $|u_i|^{2_k^\sharp} dv_g \rightharpoonup \delta_{p_0}$ weakly in the sense of measures. So $\tilde\Gamma(u_i) \to \mathcal{F}(p_0)$, a contradiction since $dist\left(\tilde\Gamma(u), \mathcal{F}(\overline{\Omega}_{\epsilon_0})\right) \geq \epsilon'$. This proves our claim.

By the Tubular Neighbourhood Theorem, the embedded submanifold $\mathcal{F}(M)$ has a tubular neighbourhood $\mathcal{U}$ in $\mathbb{R}^{2n+1}$ and there exists a smooth retraction

$$\pi : \mathcal{U} \longrightarrow \mathcal{F}(M) \tag{2.114}$$

Choose an $\epsilon_0 > 0$ small so that $\{y \in \mathbb{R}^{2n+1} : dist\,(y, \mathcal{F}(M)) < \epsilon_0\} \subset \mathcal{U}$. Then from our previous claim it follows that, there exists $\sigma_0 > 0$ such that

$$u \in \mathcal{I}_{S_k+\sigma_0} \;\Rightarrow\; \tilde\Gamma(u) \in \mathcal{U} \tag{2.115}$$

We define

$$\Gamma_M(u) = \mathcal{F}^{-1} \circ \pi \left(\int_M \mathcal{F}(x)\ |u(x)|^{2_k^\sharp}\ dv_g(x) \right) \tag{2.116}$$

Then the map $\Gamma_M : \mathcal{I}_{S_k+\sigma_0} \to M$ is continuous. Similarly as in our previous claim we have: for every $\epsilon > 0$ small there exists $\delta > 0$ such that

$$u \in \mathcal{I}_{S_k+\delta} \;\Rightarrow\; d_g\left(\Gamma_M(u), \overline{\Omega}_{\epsilon_0}\right) < \epsilon \tag{2.117}$$

Let $\pi^{\overline{\Omega}_{\epsilon_0}} : M \backslash B_{x_0}(\epsilon_0/2) \longrightarrow \overline{\Omega}_{\epsilon_0}$be a retraction. Choose an $\epsilon' > 0$ small so that $\{p \in M : d_g\left(p, \overline{\Omega}_{\epsilon_0}\right) < \epsilon'\} \subset M \backslash B_{x_0}(\epsilon_0/2)$. Then from our claim it follows that there exists a $\delta_0 > 0$ such that $\Gamma_M(u) \in M \backslash B_{x_0}(\epsilon_0/2)$ for all $u \in \mathcal{I}_{S_k+\delta_0}$. So for $u \in \mathcal{I}_{S_k+\delta_0}$ we define $\Gamma(u) := \pi^{\overline{\Omega}_{\epsilon_0}} \circ \Gamma_M(u)$. Then the map Γ satisfies the hypothesis of the proposition. This proves Proposition 2.5.3. □

Now we proceed to prove the first part of Theorem 2.1. By the regularity result obtained in Theorem 2.8.3, it is sufficient to show the existence of a non-trivial $H^2_{k,0}(\Omega_{\epsilon_0})$ weak solution to the equation (see (2.145) for the definition)

$$\begin{cases} Pu = |u|^{2_k^\sharp - 2}\, u & \text{in } \Omega_M \\ D^\alpha u = 0 & \text{on } \partial\Omega_M \quad \text{for } |\alpha| \leq k-1 \end{cases} \tag{2.118}$$

Suppose on the contrary the above equation only admits trivial solutions, we will show that this leads to a contradiction.

Definition 2.5.1. *Let $(X, \|\cdot\|)$ be a Banch space and fix $F \in C^1(X)$. A sequence (u_m) in X is a Palais-Smale sequence for F if $F(u_m) \leq C$, uniformly in m, while $DF(u_m) \to 0$ strongly in X' as $m \to +\infty$. We say that F satisfies the Palais-Smale condition at $c \in \mathbb{R}$, $(P.S)_c$ for short, if every Palais-Smale sequence (u_m) such that $F(u_m) \to c$ as $m \to +\infty$ has a strongly convergent subsequence.*

Now suppose that the functional I_P has no critical point in $\mathcal{N}_{\epsilon_0}$, that is there is not weak solution to (2.118). This is equivalent to the assertion that the functional

$$F_P(u) = \frac{1}{2} \int_{\Omega_{\epsilon_0}} uP(u)\ dv_g - \frac{1}{2_k^\sharp} \int_{\Omega_{\epsilon_0}} |u|^{2_k^\sharp}\ dv_g \tag{2.119}$$

does not admit a nontrivial critical point in $H^2_{k,0}(\Omega_{\epsilon_0})$.

Proposition 2.5.4. *If equation* (2.118) *admits only the trivial solution* $u \equiv 0$, *then the functional* I_P *satisfies the* $(P.S)_c$ *condition for* $c \in (S_k, 2^{\frac{2k}{n}} S_k)$.

Proof of Proposition 2.5.4*:* Let $(v_i) \in \mathcal{N}_{\epsilon_0}$ be a Palais-Smale sequence for the functional I_P such that $\lim\limits_{i\to+\infty} I_P(v_i) = c \in (S_k, 2^{\frac{2k}{n}} S_k)$, if this exists. Define $u_i := (I_P(v_i))^{\frac{1}{2^\sharp_k - 2}} v_i$. Then (u_i) is a Palais-Smale sequence for the functional F_P on the space $H^2_{k,0}(\Omega_{\epsilon_0})$ such that $\lim\limits_{i\to+\infty} F_P(u_i) \in (\frac{k}{n} S_k^{n/2k}, \frac{2k}{n} S_k^{n/2k})$. Since there is no nontrivial solution to (2.118), it follows from the Struwe-decomposition for polyharmonic operators by the author [**28**] that there exists $d \in \mathbb{N}$ non-trivial functions $u^j \in \mathscr{D}^{k,2}(\mathbb{R}^n)$, $j = 1, \dots, d$, such that upto a subsequence the following holds

$$F_P(u_i) = \sum_{j=1}^{d} E(u^j) + o(1) \qquad \text{as } i \to +\infty \tag{2.120}$$

where $E(u) := \frac{1}{2} \int\limits_{\mathbb{R}^n} (\Delta^{k/2} u)^2 dx - \frac{1}{2^\sharp_k} \int\limits_{\mathbb{R}^n} |u|^{2^\sharp_k} dx$. The u^j's are nontrivial solutions in $\mathscr{D}^{k,2}(\mathbb{R}^n)$ to $\Delta^k u = |u|^{2^\sharp_k - 2} u$ on $\mathbb{R}^n$ or on $\{x \in \mathbb{R}^n / x_1 < 0\}$ with Dirichlet boundary condition (we refer to [**28**] for details). It then follows from Lemma 3 and 5 of Ge-Wei-Zhou [**18**] that for any j, either u^j has fixed sign and $E(u) = \frac{k}{n} S_k^{n/2k}$, or u^j changes sign and $E(u) \geq \frac{2k}{n} S_k^{n/2k}$, contradicting $\lim\limits_{i\to+\infty} F_P(u_i) \in (\frac{k}{n} S_k^{n/2k}, \frac{2k}{n} S_k^{n/2k})$. Therefore the Palais-Smale condition holds at level $c \in (S_k, 2^{\frac{2k}{n}} S_k)$. More precisely, there is even no Palais-Smale sequence at this level. This ends the proof of Proposition 2.5.4. □

Proof of Theorem 2.1*:* By the Deformation Lemma (see Theorem II.3.11 and Remark II.3.12 in the monograph by Struwe [**33**]), there exists an retraction $\beta : \mathcal{I}_{S_k + 4\theta} \longrightarrow I^k_{S_k + \sigma_0}$, where σ_0 is as given in Proposition 2.5.3. Let $r_{\mathcal{N}_{\epsilon_0}} : H^2_{k,0}(\Omega_{\epsilon_0}) \backslash \{0\} \to \mathcal{N}_{\epsilon_0}$ be the projection given by $u \mapsto \frac{u}{\|u\|_{L^{2^\sharp_k}}}$. Consider the map $h : S^n \times [0, \iota_g/2] \to \overline{\Omega}_{\epsilon_0}$ given by

$$h(\sigma, t) := \begin{cases} \Gamma \circ \beta(r_{\mathcal{N}_{\epsilon_0}}(v^\sigma_{t,R_0})) & \text{for} \qquad t < \iota_g/2 \\ \sigma^M_{\iota_g/2} & \text{for} \qquad t = \iota_g/2 \end{cases} \tag{2.121}$$

where $\sigma^M_t := exp_{x_0}(t\sigma)$. This map is well defined and continuous by Proposition 2.5.3 and there exists $p_0 \in \overline{\Omega}_{\epsilon_0}$ such that

$$h(\sigma, t) = \begin{cases} p_0 & \text{for} \qquad t = 0 \\ exp_{x_0}(\frac{\iota_g}{2}\sigma) & \text{for} \qquad t = \iota_g/2 \end{cases} \tag{2.122}$$

So we obtain a homotopy of the embedded $(n-1)-$ dimensional sphere $\{exp_{x_0}(\frac{\iota_g}{2}\sigma) : \sigma \in S^n\}$to a point in Ω_{ϵ_0}, which is a contradiction to our topological assumption. This proves Theorem 2.1 for potentially sign-changing solutions.

2.6. Positive solutions

This section is devoted to the second part of Theorem 2.1, that is the existence of positive solutions. The proof is very similar to the proof of Theorem 2.1 with no restriction on the sign. We just stress on the specificities and refer to the proof of Theorem 2.1 everytime it is possible. We let $\Omega_M \subset M$ be any smooth $n-$dimensional submanifold of M, possibly with boundary. In the sequel, we will either take $\Omega_M = M$, or $M \backslash \overline{B_{x_0}(\epsilon_0)}$. For $u \in H^2_{k,0}(\Omega_M)$, we define $u^+ := \max\{u, 0\}$, $u^- := \max\{-u, 0\}$ and

$$\mathcal{N}_+ := \{u \in H^2_{k,0}(\Omega_M) : \int\limits_{\Omega_M} (u^+)^{2^\sharp_k} \, dv_g = 1\} \tag{2.123}$$

which is a codimension 1 submanifold of $H^2_{k,0}(\Omega_M)$. Any critical point $u \in H^2_{k,0}(\Omega_M)$ of I_g on $\mathcal{N}_+$ is a weak solution to

$$Pu = u_+^{2^\sharp_k - 1} \text{ in } \Omega_M \; ; \; D^\alpha u = 0 \text{ on } \partial\Omega_M \text{ for } |\alpha| \leq k-1. \tag{2.124}$$

Consider the Green's function G_P associated to the operator P with Dirichlet boundary condition on the smooth domain $\Omega_M \subsetneq M$, which is a function $G_P : \Omega_M \times \Omega_M \backslash \{(x,x) : x \in \Omega_M\} \longrightarrow \mathbb{R}$ such that

(i) For any $x \in \Omega_M$, the function $G_P(x, \cdot) \in L^1(\Omega_M)$

(ii) For any $\varphi \in C^\infty(\overline{\Omega_M})$ such that $D^\alpha \varphi = 0$ on $\partial\Omega_M$ for all $|\alpha| \leq k-1$, we have that

$$\varphi(x) = \int\limits_{\Omega_M} G_P(x,y) \; P\varphi(y) \; dv_g(y) \tag{2.125}$$

Lemma 2.6.1. *Let $(u_i) \in \mathcal{N}_+$ be a minimizing sequence for I^k_g on $\mathcal{N}_+$. Then*

(i) *Either there exists $u_0 \in \mathcal{N}_+$ such that $u_i \to u_0$ strongly in $H^2_{k,0}(\Omega_M)$, and u_0 is a minimizer of I_P on $\mathcal{N}_+$*

(ii) *Or there exists $x_0 \in \overline{\Omega_M}$ such that $|u_i|^{2^\sharp_k} \, dv_g \rightharpoonup \delta_{x_0}$ as $i \to +\infty$ in the sense of measures. Moreover, $\inf\limits_{u \in \mathcal{N}_+} I_P(u) = \frac{1}{K_0(n,k)}$.*

Proof of Lemma 2.6.1*:* As the functional I_g is coercive so the sequence (u_i) is bounded in $H^2_{k,0}(\Omega_M)$. We let $u_0 \in H^2_{k,0}(\Omega_M)$ such that, up to a subsequence, $u_i \rightharpoonup u_0$ weakly in $H^2_{k,0}(\Omega_M)$ as $i \to +\infty$, and $u_i(x) \to u_0(x)$ as $i \to +\infty$ for a.e. $x \in \Omega_M$. As the sequences $(u_i^+), (u_i^-)$ is bounded in $L^{2^\sharp_k}(\Omega_M)$ and $u_i^+(x) \to u_0^+(x)$, $u_i^-(x) \to u_0^-(x)$ for a.e. $x \in \Omega_M$, integration theory yields

$$u_i^+ \rightharpoonup u_0^+ \text{ and } u_i^- \rightharpoonup u_0^- \qquad \text{weakly in } L^{2^\sharp_k}(\Omega_M) \text{ as } i \to +\infty. \tag{2.126}$$

Therefore,

$$\left\|u_0^+\right\|^{2^\sharp_k}_{L^{2^\sharp_k}} \leq \liminf_{i \to +\infty} \left\|u_i^+\right\|^{2^\sharp_k}_{L^{2^\sharp_k}} = 1 \qquad \text{and} \qquad \left\|u_0^-\right\|^{2^\sharp_k}_{L^{2^\sharp_k}} \leq \liminf_{\imath \to +\infty} \left\|u_i^-\right\|^{2^\sharp_k}_{L^{2^\sharp_k}} \tag{2.127}$$

We claim that

$$u_i^- \to u_0^- \qquad \text{strongly in } L^{2^\sharp_k}(\Omega_M) \tag{2.128}$$

We prove the claim. We define $v_i := u_i - u_0$. Up to extracting a subsequence, we have that $(v_i)_i \to 0$ in $H_{k-1}^2(M)$. Therefore, as $i \to +\infty$,

$$I_P(u_i) = \int_{\Omega_M} (\Delta_g^{k/2} v_i)^2 \, dv_g + I_P(u_0) + o(1) \tag{2.129}$$

And then, letting $\alpha := \inf_{u \in \mathcal{N}_+} I_P(u)$, we have that

$$\alpha = I_P(u_i) + o(1) \geq \int_{\Omega_M} (\Delta_g^{k/2} v_i)^2 \, dv_g + \alpha \left\| u_0^+ \right\|_{L^{2_k^\sharp}}^2 + o(1)$$

and then

$$\alpha \left(1 - \left\| u_0^+ \right\|_{L^{2_k^\sharp}}^2 \right) \geq \int_{\Omega_M} (\Delta_g^{k/2} v_i)^2 \, dv_g + o(1) \tag{2.130}$$

as $i \to +\infty$. We fix $\epsilon > 0$. It then follows from (2.11) and $(v_i)_i \to 0$ in $H_{k-1}^2(M)$ that

$$\alpha \left(K_0(n,k) + \epsilon \right) \left(1 - \left\| u_0^+ \right\|_{L^{2_k^\sharp}}^2 \right) \geq \left\| v_i \right\|_{L^{2_k^\sharp}}^2 + o(1) \tag{2.131}$$

Since $(a+b)^{2_k^\sharp/2} \geq a^{2_k^\sharp/2} + b^{2_k^\sharp/2}$ for all $a, b > 0$, we get that

$$\left(\alpha \left(K_0(n,k) + \epsilon \right) \right)^{2_k^\sharp/2} \left(1 - \left\| u_0^+ \right\|_{L^{2_k^\sharp}}^{2_k^\sharp} \right) \geq \left\| v_i \right\|_{L^{2_k^\sharp}}^{2_k^\sharp} + o(1) \tag{2.132}$$

Integration theory yields $\|u_i\|_{L^{2_k^\sharp}}^{2_k^\sharp} = \|v_i\|_{L^{2_k^\sharp}}^{2_k^\sharp} + \|u_0\|_{L^{2_k^\sharp}}^{2_k^\sharp} + o(1)$ as $i \to +\infty$. Therefore

$$\begin{aligned}
&\left(\alpha \left(K_0(n,k) + \epsilon \right) \right)^{2_k^\sharp/2} \left(1 - \left\| u_0^+ \right\|_{L^{2_k^\sharp}}^{2_k^\sharp} \right) + o(1) \geq \|u_i\|_{L^{2_k^\sharp}}^{2_k^\sharp} - \|u_0\|_{L^{2_k^\sharp}}^{2_k^\sharp} \\
&= \left\| u_i^+ \right\|_{2_k^\sharp}^{2_k^\sharp} - \left\| u_0^+ \right\|_{2_k^\sharp}^{2_k^\sharp} + \left\| u_i^- \right\|_{2_k^\sharp}^{2_k^\sharp} - \left\| u_0^- \right\|_{2_k^\sharp}^{2_k^\sharp} = 1 - \left\| u_0^+ \right\|_{2_k^\sharp}^{2_k^\sharp} + \left\| u_i^- \right\|_{2_k^\sharp}^{2_k^\sharp} - \left\| u_0^- \right\|_{2_k^\sharp}^{2_k^\sharp}
\end{aligned}$$

Then $\left\| u_i^- \right\|_{L^{2_k^\sharp}}^{2_k^\sharp} = \left\| u_i^- - u_0^- \right\|_{L^{2_k^\sharp}}^{2_k^\sharp} + \left\| u_0^- \right\|_{L^{2_k^\sharp}}^{2_k^\sharp} + o(1)$ as $i \to +\infty$ yields

$$\begin{aligned}
&\left(\left(\mu \left(K_0(n,k) + \epsilon \right) \right)^{2_k^\sharp/2} - 1 \right) \left(1 - \left\| u_0^+ \right\|_{L^{2_k^\sharp}}^{2_k^\sharp} \right) + o(1) \geq \left\| u_i^- \right\|_{L^{2_k^\sharp}}^{2_k^\sharp} - \left\| u_0^- \right\|_{L^{2_k^\sharp}}^{2_k^\sharp} && (2.133) \\
&= \left\| u_i^- - u_0^- \right\|_{L^{2_k^\sharp}}^{2_k^\sharp} + o(1) && (2.134)
\end{aligned}$$

Since $\alpha K_0(n,k) \leq 1$ and $\epsilon > 0$ is arbitrary small, we get (2.128). This proves the claim.

We define $\mu_i := (\Delta_g^{k/2} u_i)^2 \, dv_g$ and $\nu_i = |u_i|^{2_k^\sharp} \, dv_g$ for all i. Up to a subsequence, we denote respectively by μ and ν their limits in the sense of measures. It follows from the concentration-compactness Theorem 2.4 that,

$$\nu = |u_0|^{2_k^\sharp} \, dv_g + \sum_{j \in \mathcal{J}} \alpha_j \delta_{x_j} \text{ and } \mu \geq (\Delta_g^{k/2} u_0)^2 \, dv_g + \sum_{j \in \mathcal{I}} \beta_j \delta_{x_i} \tag{2.135}$$

where $J \subset \mathbb{N}$ is at most countable, $(x_j)_{j\in J} \in M$ is a family of points, and $(\alpha_j)_{j\in J} \in \mathbb{R}_{\geq 0}$, $(\beta_j)_{j\in J} \in \mathbb{R}_{\geq 0}$ are such that $\alpha_j^{2/2_k^\sharp} \leq K_0(n,k)\,\beta_j$ for all $j \in J$. Since $u_i^- \to u_0^-$ strongly in $L^{2_k^\sharp}(M)$, we then get that

$$|u_i^+|^{2_k^\sharp}\,dv_g \rightharpoonup |u_0^+|^{2_k^\sharp}\,dv_g + \sum_{j\in\mathcal{J}} \alpha_j \delta_{x_j} \tag{2.136}$$

as $i \to +\infty$ in the sense of measures. The sequel is similar to the proof of Lemma 2.3.1. We omit the details. This completes the proof of Lemma 2.6.1. □

Lemma 2.6.2. *We assume that there is no nontrivial solution to* (2.124)*. Then the functional I_P satisfies the $(P.S)_c$ condition on $\mathcal{N}_+$ for $c \in (S_k, 2^{\frac{2k}{n}} S_k)$ if the equation.*

Proof of Lemma 2.6.2*:* This is equivalent to prove that the functional

$$F_P^+(u) = \frac{1}{2}\int_{\Omega_M} uP(u)\,dv_g - \frac{1}{2_k^\sharp}\int_{\Omega_M} (u^+)^{2_k^\sharp}\,dv_g \tag{2.137}$$

satisfies the $(P.S)_c$ condition on $H_{k,0}^2(\Omega_M)$ for $c \in (\frac{k}{n}S_k^{n/2k}, \frac{2k}{n}S_k^{n/2k})$. Let (u_i) be a Palais-Smale sequence for the functional F_P^+ on the space $H_{k,0}^2(\Omega_M)$. Then, as $v \in H_{k,0}^2(\Omega_M)$ goes to 0,

$$\int_{\Omega_M} u_i P_g^k(v)\,dv_g - \int_{\Omega_M} (u_i^+)^{2_k^\sharp-1} v\,dv_g = o\left(\|v\|_{H_k^2}\right) \tag{2.138}$$

Without loss of generality we can assume that $u_i \in C_c^\infty(\Omega_M)$ for all i. Let $\varphi_i \in C^\infty(\overline{\Omega}_M)$ be the unique solution of the equation

$$\begin{cases} P_g^k \varphi_i = (u_i^+)^{2_k^\sharp-1} & \text{in } \Omega_M \\ D^\alpha \varphi_i = 0 & \text{on } \partial\Omega_M \quad \text{for } |\alpha| \leq k-1 \end{cases} \tag{2.139}$$

The existence of such φ_i is guaranteed by Theorem 2.8.2. It then follows from Green's representation formula that

$$\varphi_i(x) = \int_{\Omega_M} G_P(x,y)(u_i^+(y))^{2_k^\sharp-1}\,dv_g(y) \geq 0 \tag{2.140}$$

for all $x \in \Omega_M$. Note that the sequence (φ_i) is bounded in $H_{k,0}^2(\Omega_M)$. It follows from (2.138) that $\varphi_i = u_i + o(1)$, where $o(1) \to 0$ in $H_{k,0}^2(\Omega_M)$ as $i \to +\infty$. And so (φ_i) is Palais-Smale sequences for the functional F_P^+ on the space $H_{k,0}^2(\Omega_M)$. Therefore, since $\varphi_i \geq 0$, it is also a Palais-Smale sequence for F_P defined in (2.119). Since there is no nontrivial critical point for F_P^+, using the Struwe decomposition [**28**] as in the proof of Proposition 2.5.4, we then get that $(\varphi)_i$ is relatively compact in $H_{k,0}^2(\Omega_M)$, and so is (u_i). This ends the proof of Lemma 2.6.1. □

Proof of Theorem 2.1*, positive solutions:* this goes essentially as in the proof of Theorem 2.1, the key remark being that the functions $v_{t,R}^\sigma$ defined in (2.100) are nonnegative. We define $\mathcal{N}_+^{\epsilon_0} = \{u \in H_{k,0}^2(\Omega_{\epsilon_0}) : \|u^+\|_{L^{2_k^\sharp}} = 1\}$, where $\Omega_{\epsilon_0} = M \setminus \overline{B}_{\epsilon_0}(x_0)$ and $\epsilon_0 > 0$ was defined in (2.106). For $c \in \mathbb{R}$ we define the sublevel sets of the functional I_P on $\mathcal{N}_+^{\epsilon_0}$ as $\mathcal{I}_c^+ := \{u \in \mathcal{N}_+^{\epsilon_0} : I_g^k(u) < c\}$. Arguing as in the proof of Proposition 2.5.3, it follows from Lemma 2.6.1 that there exists a $\delta_0 > 0$

such that there exists $\Gamma : \mathcal{I}^+_{S_k+\delta_0} \to \overline{\Omega}_{\epsilon_0}$ a continous map such that: If $(u_i) \in \mathcal{I}^+_{S_k+\delta_0}$ is a sequence such that $|u_i^+|^{2^\sharp_k}\, dv_g \rightharpoonup \delta_{p_0}$ weakly in the sense of measures, for some point $p_0 \in \overline{\Omega}_{\epsilon_0}$, then $\lim\limits_{i\to+\infty} \Gamma(u_i) = p_0$.

Let $r_{\mathcal{N}^{\epsilon_0}_+} : H^2_{k,0}(\Omega_{\epsilon_0})\backslash\{\left\|u^+\right\|_{L^{2^\sharp_k}} = 0\} \to \mathcal{N}^{\epsilon_0}_+$ be the map given by $u \mapsto \frac{u}{\|u^+\|_{L^{2^\sharp_k}}}$. Consider the map $h : S^n \times [0, \iota_g/2] \to \overline{\Omega}_{\epsilon_0}$ given by

$$h(\sigma,t) = \begin{cases} \Gamma \circ \beta(r_{\mathcal{N}^{\epsilon_0}_+}(v^\sigma_{t,R_0})) & \text{for} \quad t < \iota_g/2 \\ \sigma^M_{\iota_g/2} & \text{for} \quad t = \iota_g/2 \end{cases} \tag{2.141}$$

where $\beta : \mathcal{I}^+_{S_k+4\theta} \to \mathcal{I}^+_{S_k+\delta_0}$ is a retract (we have used Lemma 2.6.2) and $\sigma^M_t = exp_{x_0}(t\sigma)$. Note here that we use that $v^\sigma_{t,R_0} \geq 0$. As in the proof of Theorem 2.1, h is an homotopy of the embedded $(n-1)-$dimensional sphere $\{exp_{x_0}(\frac{\iota_g}{2}\sigma) : \sigma \in S^n\}$ to a point in Ω_{ϵ_0}, which is a contradiction to our topological assumption. So there exists a nontrivial critical point u for the functional I_P on $\mathcal{N}^{\epsilon_0}_+$, which yields a weak solution to (2.124). It then follows from the regularity theorem 2.8.3 that $u \in C^\infty(\overline{\Omega}_{\epsilon_0})$, $u > 0$, is a solution to (2.2). This ends the proof of Theorem 2.1 for positive solutions. □

2.7. An Important Remark

We remark that the topological condition of Theorem 2.1 is in general a necessary condition. Consider the n-dimensional unit sphere $\mathbb{S}^n$ endowed with its standard round metric h_r and let P_{h_r} be the conformally invariant GJMS operator on $\mathbb{S}^n$. By the stereographic projection it follows that $\mathbb{S}^n\backslash\{x_0\}$ is conformal to $\mathbb{R}^n$. Also one has that $\mathbb{S}^n\backslash\{x_0\}$ is contractible to a point. Let Ω_{ϵ_0} be the domain in $\mathbb{S}^n\backslash\{x_0\}$ constructed as earlier in (2.1), and let $u \in H^2_{k,0}(\Omega_{\epsilon_0})$, $u \neq 0$ solve the equation

$$\begin{cases} P_{h_r} u = (u^+)^{2^\sharp_k - 1} & \text{in } \Omega_{\epsilon_0} \\ D^\alpha u = 0 & \text{on } \partial\Omega_{\epsilon_0} \quad \text{for } |\alpha| \leq k-1 \end{cases} \tag{2.142}$$

Then by the stereographic projection it follows that there exists a ball of radius R, $B_0(R)$ such that there is a nontrivial solution $v \in H^2_{k,0}(B_0(R))$ to the equation

$$\begin{cases} \Delta^k v = (v^+)^{2^\sharp_k - 1} & \text{in } B_0(R) \\ D^\alpha v = 0 & \text{on } \partial B_0(R) \quad \text{for } |\alpha| \leq k-1 \end{cases} \tag{2.143}$$

By a result of Boggio[**7**], the Green's function for the Dirichlet problem above is positive. Therefore, we get that $v > 0$ is a smooth classical solution to

$$\begin{cases} \Delta^k v = v^{2^\sharp_k - 1} & \text{in } B_0(R) \\ D^\alpha v = 0 & \text{on } \partial B_0(R) \quad \text{for } |\alpha| \leq k-1 \end{cases} \tag{2.144}$$

This is impossible by Pohozaev identity, see Lemma 3 of Ge-Wei-Zhou [**18**].

2.8. Appendix: Regularity

Let $f \in L^1(\Omega_M)$. We say that $u \in H^2_{k,0}(\Omega_M)$ is a weak solution of the equation $Pu = f$ in Ω_M and $D^\alpha u = 0$ on $\partial\Omega_M$ for $|\alpha| \leq k-1$, if for all $\varphi \in C_c^\infty(\Omega_M)$

$$\int_{\Omega_M} \Delta_g^{k/2} u \, \Delta_g^{k/2} \varphi \, dv_g + \sum_{\alpha=0}^{k-1} \int_{\Omega_M} A_l(g)(\nabla^l u, \nabla^l \varphi) \, dv_g = \int_{\Omega_M} f\varphi \, dv_g \tag{2.145}$$

Now let the operator P be coercive on the space $H^2_{k,0}(\Omega_M)$, i.e there exists a constant $C > 0$ such that for all $u \in H^2_{k,0}(\Omega_M)$

$$\int_{\Omega_M} uP(u) \, dv_g \geq \; C \|u\|^2_{H^2_{k,0}(\Omega_M)} . \tag{2.146}$$

We then have

Proposition 2.8.1 ((H^p_k-coercivity)**.**

$$\inf_{u \in H^p_k(\Omega_M)\setminus\{0\}} \frac{\|Pu\|_p}{\|u\|_{H^p_k}} > 0 \tag{2.147}$$

Proof of Proposition 2.8.1*:* We proceed by contradiction. If not, then there exists a sequence $(u_i) \in C_c^\infty(\Omega_M)$ such that $\|u_i\|_{H^p_k} = 1$ and $\lim\limits_{i \to +\infty} \|Pu_i\|_p = 0$. It follows from classical estimates (see Agmon-Douglis-Nirenberg [**1**]) that

$$\|u_i\|_{H^p_{2k}(\Omega_M)} \leq C_p \left(\|Pu_i\|_{L^p} + \|u_i\|_{H^p_k} \right) = O(1) \tag{2.148}$$

So there exists $u_0 \in H^p_{2k,0}(\Omega_M)$ such that upto a subsequence $u_i \rightharpoonup u_0$ weakly in $H^p_{2k,0}(\Omega_M)$. Then $u_i \to u_0$ strongly in $H^p_{k,0}(\Omega_M)$ and so $\|u_0\|_{H^p_k} = 1$. Also u_0 weakly solves the equation $Pu_0 = 0$ in Ω_M and $D^\alpha u_0 = 0$ on $\partial\Omega_M$ for $|\alpha| \leq k-1$. It follows from standard elliptic estimates (see Agmon-Douglis-Nirenberg [**1**]) that $u_0 \in C^\infty(\overline{\Omega}_M)$. Then, multiplying the equation by u_0 and integrating over M, coercivity yields

$$C \|u_0\|^2_{H^2_k(\Omega_M)} \leq \int_M u_0 P u_0 \, dv_g = 0 \tag{2.149}$$

and hence $u_0 \equiv 0$, a contradiction since we have also obtained that $\|u_0\|_{H^p_k} = 1$. This proves Proposition 2.8.1. □

Proposition 2.8.2 (Existence and Uniqueness)**.** *Let the operator P^k_g be coercive. Then given any $f \in L^p(\Omega_M)$, $1 < p < +\infty$, there exists a unique weak solution $u \in H^p_{k,0}(\Omega_M) \cap H^p_{2k}(\Omega_M)$ to*

$$\begin{cases} Pu = f & \text{in } \Omega_M \\ D^\alpha u = 0 & \text{on } \partial\Omega_M \quad \text{for } |\alpha| \leq k-1 \end{cases} \tag{2.150}$$

The proof is classical and we only sketch it here. For $p = 2$, existence and uniqueness follows from the Riesz representation theorem in Hilbert spaces. For arbitrary $p > 1$, we approximate f in L^p by smooth compactly supported function on Ω_M. For each of these smooth functions, there exists a solution to the pde with the approximation as a right-hand-side. The coercivity and the Agmon-Douglis-Nirenberg estimates yield convergence of these solutions to a solution of the original equation. Coercivity yields uniqueness.

We now proceed to prove our regularity results. The proof is based on ideas developed by Van der Vorst [**35**], and also employed by Djadli-Hebey-Ledoux [**13**] for the case $k = 2$.

Theorem 2.5. *Let (M, g) be a smooth, compact Riemannian manifold of dimension n and let k be a positve integer such that $2k < n$. Let Ω_M be a smooth domain in M and suppose $u \in H^2_{k,0}(\Omega_M)$ be a weak solution of the equation*

$$\begin{cases} Pu = f(x,u) & \text{in } \Omega_M \\ D^\alpha u = 0 & \text{on } \partial\Omega_M \quad \text{for } |\alpha| \le k-1 \end{cases} \tag{2.151}$$

where $|f(x,u)| \le C|u|(1 + |u|^{2^\sharp_k - 2})$ for some positive constant C, then

$$u \in L^p(\Omega_M) \qquad \text{for all } \; 1 < p < +\infty \tag{2.152}$$

Proof of 2.5*:* We write $f(x,u) = bu$ where $|b| \le C(1 + |u|^{2^\sharp_k - 2})$. Then $b \in L^{n/2k}(\Omega_M)$ and u solves weakly the equation

$$\begin{cases} Pu = bu & \text{in } \Omega_M \\ D^\alpha u = 0 & \text{on } \partial\Omega_M \quad \text{for } |\alpha| \le k-1 \end{cases} \tag{2.153}$$

Step 1: We claim that for any $\epsilon > 0$ there exists $q_\epsilon \in L^{n/2k}(\Omega_M)$ and $f_\epsilon \in L^\infty(\Omega_M)$ such that

$$bu = q_\epsilon u + f_\epsilon, \qquad \|q_\epsilon\|_{L^{n/2k}(\Omega_M)} < \epsilon \tag{2.154}$$

Now $\lim\limits_{i \to +\infty} \int\limits_{\{|u| \ge i\}} |b|^{n/2k} \, dv_g = 0$, so given any $\epsilon > 0$ we can choose i_0 such that

$$\int\limits_{\{|u| \ge i_0\}} |b|^{n/2k} \, dv_g < \epsilon^{n/2k}.$$

We define $q_\epsilon := \chi_{\{|u| \ge i_0\}} b$ and $f_\epsilon := (b - q_\epsilon)u = \chi_{\{|u| < i_0\}} b$. Then, since $|b| \le C(1 + |u|^{2^\sharp_k - 2})$, we have that $\|q_\epsilon\|_{L^{n/2k}(\Omega_M)} < \epsilon$ and $f_\epsilon \in L^\infty(M)$. This proves our claim and ends Step 1.

We rewrite (2.153) as

$$\begin{cases} Pu = q_\epsilon u + f_\epsilon & \text{in } \Omega_M \\ D^\alpha u = 0 & \text{on } \partial\Omega_M \quad \text{for } |\alpha| \le k-1 \end{cases} \tag{2.155}$$

Let $\mathscr{H}_\epsilon$ be the operator defined formally as

$$\mathscr{H}_\epsilon u = (P^k_g)^{-1}(q_\epsilon u) \tag{2.156}$$

Then $Pu = q_\epsilon u + f_\epsilon$ becomes $u - \mathscr{H}_\epsilon u = (P^k_g)^{-1}(f_\epsilon)$.

Step 2: we claim that for any $s > 1$, $\mathscr{H}_\epsilon$ maps $L^s(\Omega_M)$ to $L^s(\Omega_M)$.

We prove the claim. Let $v \in L^s(\Omega_M)$, $s \geq 2_k^\sharp$, then $q_\epsilon v \in L^{\hat{s}}(\Omega_M)$ where $\hat{s} = \dfrac{ns}{n+2ks}$, and we have by Hölder inequality

$$\|q_\epsilon v\|_{L^{\hat{s}}(\Omega_M)} \leq \|q_\epsilon\|_{L^{n/2k}(\Omega_M)} \|v\|_{L^s(\Omega_M)} \tag{2.157}$$

Since $\|q_\epsilon\|_{L^{n/2k}(\Omega_M)} < \epsilon$, so we have

$$\|q_\epsilon v\|_{L^{\hat{s}}(\Omega_M)} \leq \epsilon \|v\|_{L^s(\Omega_M)} \tag{2.158}$$

From (2.8.2) it follows that there exists a unique $v_\epsilon \in H_{2k}^{\hat{s}}(\Omega_M)$ such that

$$\begin{cases} Pv_\epsilon = q_\epsilon v & \text{in } \Omega_M \\ D^\alpha v_\epsilon = 0 & \text{on } \partial\Omega_M \quad \text{for } |\alpha| \leq k-1 \end{cases} \tag{2.159}$$

weakly. Further we have for a positive constant $C(s)$

$$\|v_\epsilon\|_{H_{2k}^{\hat{s}}(\Omega_M)} \leq C(s) \|q_\epsilon v\|_{L^{\hat{s}}(\Omega_M)} \tag{2.160}$$

So we obtained that

$$\|v_\epsilon\|_{H_{2k}^{\hat{s}}(\Omega_M)} \leq C(s)\epsilon \|v\|_{L^s(\Omega_M)} \tag{2.161}$$

By Sobolev embedding theorem $H_{2k}^{\hat{s}}(\Omega_M)$ is continuously imbedded in $L^s(\Omega_M)$ so $v_\epsilon \in L^s(\Omega_M)$ and we have

$$\|v_\epsilon\|_{L^s(\Omega_M)} \leq C(s)\epsilon \|v\|_{L^s(\Omega_M)} \tag{2.162}$$

In other words, for any $s \geq 2_k^\sharp$ the operator $\mathscr{H}_\epsilon$ acts from $L^s(\Omega_M)$ into $L^s(\Omega_M)$, and its norm $\|\mathscr{H}_\epsilon\|_{L^s \to L^s} \leq C(s)\epsilon$. This proves the claim and ends Step 2.

Step 3: Now let $s \geq 2_k^\sharp$ be given, then for $\epsilon > 0$ sufficiently small one has

$$\|\mathscr{H}_\epsilon\|_{L^s \to L^s} \leq \frac{1}{2} \tag{2.163}$$

and so the operator $I - \mathscr{H}_\epsilon : L^s(\Omega_M) \longrightarrow L^s(\Omega_M)$ is invertible. We have

$$u - \mathscr{H}_\epsilon u = (P_g^k)^{-1}(f_\epsilon) \tag{2.164}$$

Since $u \in L^{2_k^\sharp}(\Omega_M)$ and $f_\epsilon \in L^\infty(\Omega_M)$, so $u \in L^p(\Omega_M)$ for all $1 < p < +\infty$.

This ends the proof of Theorem 2.5. □

Proposition 2.8.3. *Let (M, g) be a smooth, compact Riemannian manifold of dimension n and let k be a positive integer such that $2k < n$. Let $f \in C^{0,\theta}(\Omega_M)$ a Hölder continuous function. Let Ω_M be a smooth domain in M and suppose $u \in H_{k,0}^2(\Omega_M)$ be a weak solution of the equation*

$$\begin{cases} Pu = f\,|u|^{2_k^\sharp - 2} u \ \textit{or}\ f(u^+)^{2_k^\sharp - 1} & \textit{in } \Omega_M \\ D^\alpha u = 0 & \textit{on } \partial\Omega_M \quad \textit{for } |\alpha| \leq k-1 \end{cases} \tag{2.165}$$

Then $u \in C^{2k}(\Omega_M)$, and is a classical solution of the above equation. Further if $u > 0$ and $f \in C^\infty(\Omega_M)$, then $u \in C^\infty(\Omega_M)$.

Proof of Proposition 2.8.3*:* It follows from (2.5) that $u \in H^p_{2k}(\Omega_M)$ for all $1 < p < +\infty$. By Sobolev imbedding theorem this implies $u \in C^{2k-1,\gamma}(\overline{\Omega}_M)$ for all $0 < \gamma < 1$. $|u|^{2^\sharp_k - 2}\, u, (u^+)^{2^\sharp_k - 1} \in C^1(\overline{\Omega}_M)$. The Schauder estimates (here again, we refer to Agmon-Douglis-Nirenberg [**1**]) then yield $u \in C^{2k,\gamma}(\overline{\Omega}_M)$ for all $\gamma \in (0,1)$, and u is a classical solution.
If $u > 0$, then the right-hand-side is $u^{2^\sharp_k - 1}$ and has the same regularity as u. Therefore, iterating the Schauder estimates yields $u \in C^\infty(\overline{\Omega_M})$. This ends the proof of Proposition 2.8.3. □

2.9. Appendix: Local Comparison of the Riemannian norm with the Euclidean norm

Let (M,g) be a smooth, compact Riemannian manifold of dimension $n \geq 1$. For any point $p \in M$ there exists a local coordinate around p, $\varphi_p^{-1} : \Omega \subset \mathbb{R}^n \to M$, $\varphi(p) = 0$, such that in these local coordinates one has for all indices $i,j,k = 1,\dots,n$

$$\begin{cases} (1-\epsilon)\delta_{ij} \leq g_{ij}(x) \leq (1+\epsilon)\delta_{ij} & \text{as bilinear forms.} \\ |g_{ij}(x) - \delta_{ij}| \leq \epsilon \end{cases}$$

Here we have identified $T_pM \cong \mathbb{R}^n$ for any point $p \in M$. For example, one can take the exponetial map at $p : exp_p$, which is normal at p. We will let ι_g be the injectivity radius of M. Using the above local comparison of the Riemannian metric with the Euclidean metric one obtains

Lemma 2.9.1. *Let (M,g) be a smooth, compact Riemannian manifold of dimension n and let k be positive integer such that $2k < n$. We fix $s \geq 1$. Let $\varphi_p^{-1} : \Omega \subset \mathbb{R}^n \to M$, $\varphi(p) = 0$ be a local coordinate around p with the above mentioned properties. Then given any $\epsilon_0 > 0$ there exists $\tau \in (0, \iota_g)$, such that for any point $p \in M$, and $u \in C_c^\infty(B_0(\tau))$ one has*

(2.166)
$$(1-\epsilon_0)\int_{\mathbb{R}^n} (\Delta^{k/2} u)^2\, dx \leq \int_M (\Delta_g^{k/2}(u \circ \varphi_p))^2\, dv_g \leq (1+\epsilon_0)\int_{\mathbb{R}^n} (\Delta^{k/2} u)^2\, dx$$

and

$$(1-\epsilon_0)\int_{\mathbb{R}^n} |u|^s\, dx \leq \int_M |u \circ \varphi_p|^s\, dv_g \leq (1+\epsilon_0)\int_{\mathbb{R}^n} |u|^s\, dx \tag{2.167}$$

Proof of Lemma 2.9.1*:* In terms of the coordinate map $\varphi_p^{-1} : \Omega \subset \mathbb{R}^n \to M$, for any $f \in C^2(M)$ we have

$$\Delta_g f\,(x) = -g^{ij}(x)\left(\frac{\partial^2 (f \circ \varphi^{-1})}{\partial x_i \partial x_j}(x) - \Gamma_{ij}^k(\varphi(x))\frac{\partial (f\circ\varphi^{-1})}{\partial x_k}(x)\right). \tag{2.168}$$

Since the manifold M is compact, then given any $\epsilon > 0$ there exists a $\tau \in (0, \iota_g)$ depending only on (M,g), such that for any point $p \in M$ and for any $x \in B_0(\tau) \subset \mathbb{R}^n$ one has for all indices $i,j,k = 1,\dots,n$

$$\begin{cases} (1-\epsilon)\delta_{ij} \leq g_{ij}(x) \leq (1+\epsilon)\delta_{ij} & \text{as bilinear forms.} \\ |g_{ij}(x) - \delta_{ij}| \leq \epsilon \end{cases}$$

Without loss of generality we can assume that $\tau < 1$. We let $u \in C_c^\infty(\mathbb{R}^n)$ be such that $supp(u) \subset B_0(\tau)$. In the sequel, the constant C will denote any positive constant depending only on (M,g) and τ: the same notation C may apply to different constants from line to line, and even in the same line. All integrals below are taken over $B_0(\tau)$, and we will therefore omit to write the domain for the sake of clearness.

Case 1: k is even. We then write $k=2m$, $m\geq 1$. Then calculating in terms of local coordinates we obtain

$$\left|\Delta_g^m(u\circ\varphi_p)(\varphi_p^{-1}(x))-\Delta^m u(x)\right|\leq\epsilon\left|\nabla^{2m}u(x)\right|+C\sum_{\beta=1}^{2m-1}\left|\nabla^{(2m-\beta)}u(x)\right| \tag{2.169}$$

where C_g is a constant depending only on the metric g on M. Then we have

$$\left|\int\left(\Delta_g^m(u\circ\varphi_p)(\varphi_p^{-1}(x))\right)^2\,dx-\int(\Delta^m u)^2\,dx\right|\leq 2^2\epsilon^2\int\left|\nabla^{2m}u\right|^2dx+ \tag{2.170}$$

$$C\sum_{\beta=1}^{2m-1}\int\left|\nabla^{(2m-\beta)}u\right|^2dx+2\epsilon\int\left|\nabla^{2m}u\right|\left|\Delta^m u\right|dx+C\sum_{\beta=1}^{2m-1}\int\left|\Delta^m u\right|\left|\nabla^{(2m-\beta)}u\right|dx \tag{2.171}$$

Now for any β such that $\beta\leq 2m-1$ we have $\nabla^{(2m-\beta)}u\in\mathscr{D}^{\beta,2}(\mathbb{R}^n)$ and by Sobolev embedding theorem this implies that $\left|\nabla^{(2m-\beta)}u\right|^2\in L^{2^\sharp_\beta/2}(\mathbb{R}^n)$. Applying the Hölder inequality we obtain

$$\sum_{\beta=1}^{2m-1}\int\left|\nabla^{(2m-\beta)}u\right|^2dx\leq C\left(\sum_{\beta=1}^{2m-1}\tau^{2\beta}\right)\left(\int\left|\nabla^{(2m-\beta)}u\right|^{2^\sharp_\beta}dx\right)^{2/2^\sharp_\beta} \tag{2.172}$$

And then the Sobolev inequality gives us

$$\left(\int\left|\nabla^{(2m-\beta)}u\right|^{2^\sharp_\beta}dx\right)^{2/2^\sharp_\beta}\leq C\int\left|\nabla^{2m}u\right|^2dx \tag{2.173}$$

Applying the integration by parts formula, we obtain

$$\int\left|\nabla^{2m}u\right|^2dx=\int(\Delta^m u)^2\,dx \tag{2.174}$$

So we have, since $\tau<1$

$$\sum_{|\beta|=1}^{2m-1}\int\left|\nabla^{2m-\beta}u\right|^2dx\leq C\tau\int(\Delta^m u)^2\,dx \tag{2.175}$$

Therefore, we get that

$$\left|\int\left(\Delta_g^m(u\circ\varphi_p)(\varphi_p^{-1}(x))\right)^2\,dx-\int(\Delta^m u)^2\,dx\right|\leq C\left(\epsilon+\tau\right)\int(\Delta^m u)^2\,dx \tag{2.176}$$

Now in these local coordinates one has

$$(1-\epsilon)^{n/2}\int \left(\Delta_g^m(u\circ\varphi_p)(\varphi_p^{-1}(x))\right)^2\ dx \leq \int\limits_M \left(\Delta_g^m(u\circ\varphi_p)\right)^2\ dv_g \tag{2.177}$$

$$\leq (1+\epsilon)^{n/2}\ \int \left(\Delta_g^m(u\circ\varphi_p)(\varphi_p^{-1}(x))\right)^2\ dx \tag{2.178}$$

So given an $\epsilon_0>0$ small, we first choose ϵ small and then choose a sufficiently small τ, so that for any $u\in C_c^\infty\left(B_0(\tau)\right)$ we have

$$\left|\int \left(\Delta_g^m(u\circ\varphi_p)\right)^2\ dv_g - \int \left(\Delta^m u\right)^2 dx\right| \leq \epsilon_0\int \left(\Delta^m u\right)^2 dx \tag{2.179}$$

So we have the lemma for k even.

Case 2: k is odd. We then write $k=2m+1$ with $m\geq 0$. Calculating in terms of local coordinates, like in the even case, we obtain

$$\left|\ |\nabla(\Delta_g^m(u\circ\varphi_p))|^2(\varphi_p^{-1}(x)) - |\nabla(\Delta^m u)|^2(x)\right| \leq \epsilon|\nabla(\Delta^m u)|^2(x) \tag{2.180}$$

$$+C\epsilon\left|\nabla^{2m+1}u\right|^2(x) + C\sum_{\beta=1}^{2m}\left|\nabla^{(2m+1-\beta)}u\right|^2(x) \tag{2.181}$$

$$+C\epsilon\left|\nabla^{2m+1}u\right|(x)\ |\nabla(\Delta^m u)|(x) + C\sum_{\beta=1}^{2m}\left|\nabla^{(2m+1-\beta)}u\right|(x)\ |\nabla(\Delta^m u)|(x) \tag{2.182}$$

for all $x\in B_0(\tau)$. Therefore

$$\begin{aligned}&\left|\int |\nabla(\Delta_g^m(u\circ\varphi_p))|^2(\varphi_p^{-1}(x))\ dx - \int |\nabla(\Delta^m u)|^2(x)\ dx\right| \leq \epsilon\int |\nabla(\Delta^m u)|^2\ dx\\ &+C\epsilon\int \left|\nabla^{2m+1}u\right|^2\ dx + C\sum_{\beta=1}^{2m}\int\left|\nabla^{(2m+1-\beta)}u\right|^2\ dx\\ &+C\epsilon\int \left|\nabla^{2m+1}u\right| |\nabla(\Delta^m u)|\ dx + C\sum_{\beta=1}^{2m}\int\left|\nabla^{(2m+1-\beta)}u\right| |\nabla(\Delta^m u)|\ dx\end{aligned} \tag{2.183}$$

And then by calculations similar to the even case, along with the integration by parts formula, we obtain

$$\left|\int |\nabla(\Delta_g^m(u\circ\varphi_p))|^2(\varphi_p^{-1}(x))\ dx - \int |\nabla(\Delta^m u)|^2(x)\ dx\right| \leq \tilde{C}_g\left(\epsilon+\sqrt{\tau}\right)\int |\nabla(\Delta^m u)|^2\ dx \tag{2.184}$$

Now given an $\epsilon_0>0$ small, we first choose ϵ small and then choose a sufficiently small τ, so that for any $u\in C_c^\infty\left(B_0(\tau)\right)$ we have

$$\left|\int\limits_M |\nabla(\Delta_g^m(u\circ\varphi_p))|^2\ dv_g - \int |\nabla(\Delta^m u)|^2\ dx\right| \leq \epsilon_0\int |\nabla(\Delta^m u)|^2\ dx \tag{2.185}$$

Then one has the lemma for k odd. This ends the proof of Lemma 2.9.1. $\square$

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CHAPTER 3

Struwe's decomposition for a Polyharmonic Operator on a compact Riemannian manifold with or without boundary

ABSTRACT. Given a high-order elliptic operator on a compact manifold with or without boundary, we perform the decomposition of Palais-Smale sequences for a nonlinear problem as a sum of bubbles. This is a generalization of the celebrated 1984 result of Struwe [**16**]. Unlike the case of second-order operators, bubbles close to the boundary might appear. Our result includes the case of a smooth bounded domain of $\mathbb{R}^n$.

3.1. Introduction

Let (M, g) be a smooth, compact Riemannian manifold of dimension n with or without boundary. In the latter case we understand that $\overline{M}$ is a compact, oriented submanifold of $(\tilde{M}, g)$ which is itself a smooth, compact Riemannian manifold without boundary and with the same metric g and dimension n. As one checks, this includes smooth bounded domains of $\mathbb{R}^n$. When the boundary $\partial M \neq \emptyset$, we let ν be its outward oriented normal vector in $\tilde{M}$. Let k be a positive integer such that $2k < n$. We define the Sobolev space $H^2_{k,0}(M)$ as the completion of $C^\infty_c(M)$ for the norm $u \mapsto \sum_{i=0}^k \|\nabla^i u\|_2$. This norm is equivalent (see Robert [**14**]) to the Hilbert norm $\|u\|_{H^2_k} := \left(\sum_{l=0}^k \int_M (\Delta_g^{l/2} u)^2 \, dv_g\right)^{1/2}$ where $\Delta_g := -\mathrm{div}_g(\nabla)$ is the Laplace-Beltrami operator and, for α odd, $\Delta_g^\alpha u \Delta_g^\alpha v := (\nabla \Delta_g^{\frac{\alpha-1}{2}} u, \nabla \Delta_g^{\frac{\alpha-1}{2}} v)_g$ for all $u, v \in H^2_k(M)$. For details we refer to Aubin [**3**] and Hebey [**9**].

We consider the functional

$$I(u) := \frac{1}{2} \int_M (\Delta_g^{k/2} u)^2 \, dv_g + \frac{1}{2} \sum_{l=0}^{k-1} \int_M A_l(\nabla^l u, \nabla^l u) \, dv_g - \frac{1}{2^\sharp_k} \int_M |u|^{2^\sharp_k} \, dv_g$$

where for all $l \in \{0, \dots, k-1\}$, A_l is a smooth T^0_{2l}-tensor field on M and A_l is symmetric (that is $A_l(X, Y) = A_l(Y, X)$ for all T^l_0-tensors X, Y on M). Here, $2^\sharp_k := \frac{2n}{n-2k}$ is the critical Sobolev exponent such that $H^2_{k,0}(M) \hookrightarrow L^{2^\sharp_k}(M)$ is continuous, which makes the definition of I consistent for all $u \in H^2_{k,0}(M)$. Critical points $u \in H^2_{k,0}(M)$ for I are weak solutions to the pde

$$\left\{ \begin{array}{lll} Pu = |u|^{2^\sharp_k - 2} u & \text{in } M & \\ \partial^\alpha_\nu u = 0 & \text{on } \partial M & \text{for } |\alpha| \leq k-1 \end{array} \right. \tag{3.1}$$

where for any $u \in C^{2k}(M)$, we define

$$Pu := \Delta_g^k u + \sum_{l=0}^{k-1} (-1)^l \nabla^{j_l \dots j_1} \left((A_l)_{i_1 \dots i_l, j_1 \dots j_l} \nabla^{i_1 \dots i_l} u \right)$$

and where we say that $u \in H^2_{k,0}(M)$ is a weak solution to (3.1) if

$$\int_M \Delta_g^{k/2} u, \Delta_g^{k/2} \varphi \, dv_g + \sum_{l=0}^{k-1} \int_M A_l(\nabla^l u \nabla^l \varphi) \, dv_g = \int_M |u|^{2_k^\sharp - 2} u \varphi \, dv_g$$

for all $\varphi \in H^2_{k,0}(M)$. As shown by the regularity theorem in Mazumdar [**13**], a weak solution u to (3.1) is indeed a strong solution, $u \in C^{2k}(\overline{M})$.

Definition 3.1.1. *Let $(X, \|\cdot\|)$ be a Banach space and $F \in C^1(X)$. A sequence (u_α) in X is said to be a Palais-Smale sequence for F if $(F(u_\alpha))_\alpha$ has a limit in $\mathbb{R}$ when $\alpha \to +\infty$, while $DF(u_\alpha) \to 0$ strongly in X' as $\alpha \to +\infty$.*

In this chapter, we describe the lack of relative compactness of Palais-Smale sequences for I, which is due to the noncompact embedding $H^2_{k,0}(M) \hookrightarrow L^{2_k^\sharp}(M)$. For Ω any open domain of $\mathbb{R}^n$, we let $\mathcal{D}^2_k(\Omega)$ be the completion of $C_c^\infty(\Omega)$ for the norm $u \mapsto \|\Delta^{k/2} u\|_2$. The limiting equations of (3.1) are

$$\Delta^k u = |u|^{2_k^\sharp - 2} u \text{ in } \mathbb{R}^n, \; u \in \mathcal{D}^2_k(\mathbb{R}^n) \tag{3.2}$$

$$\left\{ \begin{array}{ll} \Delta^k u = |u|^{2_k^\sharp - 2} u & \text{in } \mathbb{R}^n_- \\ \partial_\nu^\alpha u = 0 & \text{on } \partial\mathbb{R}^n_- \end{array} \right\}, \; u \in \mathcal{D}^2_k(\mathbb{R}^n_-) \tag{3.3}$$

where $\Delta := \Delta_{\text{Eucl}}$ is the Laplacian on $\mathbb{R}^n$ (with the minus sign convention) endowed with the Euclidean metric Eucl. Associated to the functional I is the limiting functional

$$E(u) := \frac{1}{2} \int_{\mathbb{R}^n} (\Delta^{k/2} u)^2 \, dx - \frac{1}{2_k^\sharp} \int_{\mathbb{R}^n} |u|^{2_k^\sharp} \, dx \text{ for all } u \in \mathcal{D}^2_k(\mathbb{R}^n).$$

Our main theorem below shows that the lack of convergence to a solution of equation (3.1) is described by a sum of Bubbles:

Theorem 3.1. *Let (u_α) be a Palais-Smale sequence for the functional I on the space $H^2_{k,0}(M)$. Then there exists $d \in \mathbb{N}$ bubbles $[(x_\alpha^{(j)}), (r_\alpha^{(j)}), u^{(j)}]$, $j = 1, ..., d$, (see Definition 3.2.1 below) there exists $u_\infty \in H^2_{k,0}(M)$ a solution to (3.1) such that, up to a subsequence,*

$$u_\alpha = u_\infty + \sum_{j=1}^d B_{x_\alpha^{(j)}, r_\alpha^{(j)}}(u^{(j)}) + o(1) \text{ where } \lim_{\alpha \to +\infty} o(1) = 0 \text{ in } H^2_{k,0}(M)$$

and

$$I(u_\alpha) = I(u_\infty) + \sum_{j=1}^d E(u^{(j)}) + o(1) \quad \text{as } \alpha \to +\infty.$$

In Section 3.2, Bubbles are defined up to a term going to 0 strongly, which is relevent here. As one checks, given $u \in \mathcal{D}^2_k(\mathbb{R}^n)$ a nontrivial weak solution to (3.2) or (3.3), then multiplying the equation by u and integrating by parts yields

$$E(u) \geq \beta^\sharp := \frac{k}{n} K_0(n,k)^{-n/2k} \tag{3.4}$$

where $K_0(n,k)$ be the best constant of the embedding $\mathcal{D}_k^2(\mathbb{R}^n) \hookrightarrow L^{2_k^\sharp}(\mathbb{R}^n)$, that is

$$K_0(n,k)^{-1} = \inf_{u \in \mathcal{D}_k^2(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n} (\Delta^{k/2} u)^2\, dx}{\left(\int_{\mathbb{R}^n} |u|^{2_k^\sharp}\, dx\right)^{\frac{2}{2_k^\sharp}}} \tag{3.5}$$

When the Palais-Smale sequence is nonnegative, the bubbles are positive and correspond to positive solutions to (3.2). As shown in Lions [**12**], Swanson [**17**], Ge-Wei-Zhou [**7**], these solutions are exactly the extremals for (3.5) and are of the form

$$u = U_{a,\lambda} := \alpha_{n,k} \left(\frac{\lambda}{1+\lambda^2|\cdot - a|^2}\right)^{\frac{n-2k}{2}} \quad a \in \mathbb{R}^n, \lambda > 0 \tag{3.6}$$

where $\alpha_{n,k} > 0$ is explicit. We then get the following:

Theorem 3.2. *Let (u_α) be a Palais-Smale sequence for the functional I on the space $H_{k,0}^2(M)$. We assume that $u_\alpha \geq 0$ for all $\alpha \in \mathbb{N}$. Then there exists $u_\infty \in H_{k,0}^2(M)$ a solution to* (3.1)*, there exists $d \in \mathbb{N}$ sequences : $(x_\alpha^{(1)}), \dots, (x_\alpha^{(d)}) \in M$, $(r_\alpha^{(1)}), \dots, (r_\alpha^{(d)}) \in (0,+\infty)$ such that $r_\alpha^{(j)} \to 0$ and $r_\alpha^{(j)} = o(d(x_\alpha^{(j)}, \partial M))$ as $\alpha \to +\infty$ for all $j = 1, ..., d$, and up to a subsequence,*

$$u_\alpha = u_\infty + \sum_{j=1}^{d} \eta\left((\tilde{r}_\alpha^{(j)})^{-1} exp_{x_\alpha^{(j)}}^{-1}(\cdot)\right) \alpha_{n,k} \left(\frac{r_\alpha^{(j)}}{(r_\alpha^{(j)})^2 + d_g(\cdot, x_\alpha^{(j)})^2}\right)^{\frac{n-2k}{2}} + o(1)$$

where $\lim_{\alpha\to+\infty} o(1) = 0$ in $H_{k,0}^2(M)$, and η and $(\tilde{r}_\alpha^{(j)})'s$ are as in (3.8)*. Moreover,*

$$I(u_\alpha) = I(u_\infty) + d\beta^\sharp + o(1) \quad as\ \alpha \to +\infty$$

where $\beta^\sharp$ is as in (3.4)*.*

When $k = 1$ and M is a smooth bounded domain of $\mathbb{R}^n$, Theorem 3.1 is the pioneering result of Struwe [**16**]. There have been several extensions. Without being exhaustive, we refer to Hebey-Robert [**11**] for $k = 2$ and manifolds without boundary, Saintier [**15**] for the $p-$Laplace operator, El-Hamidi-Vétois [**5**] for anisotropic operators and Almaraz [**1**] for nonlinear boundary conditions. When the manifold is the entire flat space $\mathbb{R}^n$, the decomposition is in the monograph by Fieseler-Tintarev [**18**]. Another possible description is in the sense of measures as in Lions [**12**]: a general result of this flavour for high order elliptic operators on manifolds is in Mazumdar [**13**].

Palais-Smale sequence are produced via critical point techniques, like the Mountain-Pass Lemma of Ambrosetti-Rabinowitz [**2**] or other topological methods (see for instance the monograph Ghoussoub [**8**] and the references therein). Concerning higher-order problems, we refer to Bartsch-Weth-Willem [**4**], Ge-Wei-Zhou [**7**], Mazumdar [**13**], the general monograph Gazzola-Grunau-Sweers [**6**] and the references therein. Theorem 3.1 is used by the author in [**13**] to get Coron-type solutions to equation (3.1).

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3.2. Definition of Bubbles

In the spirit of the exponential map, we first cook up a chart around any boundary point. We fix $x_0 \in \partial M$. Since M is a smooth submanifold of $\tilde{M}$, there exist Ω an open subset of $\tilde{M}$ with $x_0 \in \Omega$, there exists $U \subset \mathbb{R}^n$ open with $0 \in U$, such that for any $x \in \Omega \cap \partial M$ there exists $\mathcal{T}_x \in C^\infty(U, \tilde{M})$ having the following properties.

$$
\begin{cases}
\bullet \quad \mathcal{T}_x(0) = x \\
\bullet \quad \mathcal{T}_x \text{ is a smooth diffeomorphism onto its image } \mathcal{T}_x(U). \\
\bullet \quad \mathcal{T}_x\left(U \cap \{x_1 < 0\}\right) = \mathcal{T}_x(U) \cap M \\
\bullet \quad \mathcal{T}_x\left(U \cap \{x_1 = 0\}\right) = \mathcal{T}_x(U) \cap \partial M \\
\bullet \quad (x,z) \mapsto \mathcal{T}_x(z) \text{ is smooth from } \Omega \times U \text{ to } \tilde{M} \\
\bullet \quad d\mathcal{T}_x(0) : \mathbb{R}^n \to T_x M \text{ is an isometry} \\
\bullet \quad d\mathcal{T}_x(0)[e_1] = \nu_x \text{ where } \nu_x \text{ is the outer unit normal vector to } \partial M \\
\quad\ \ \text{at the point } x.
\end{cases} \tag{3.7}
$$

This map is defined uniformly with respect to x in a neighborhood Ω of a fixed point $x_0 \in \partial M$. By a standard abuse of notation, we will always consider $x \mapsto \mathcal{T}_x$ without any reference to Ω or x_0: this will make sense in the sequel since the relevant points will always be in the neighborhood of a fixed point.

Definition 3.2.1. *A "Bubble" is a triplet $[(x_\alpha), (r_\alpha), u]$ where $x_\alpha \in \overline{M}$ is a convergent sequence, $r_\alpha > 0$ for all $\alpha \in \mathbb{N}$ with $\lim_{\alpha\to+\infty} r_\alpha = 0$ and*

$$
\text{either } \left\{ x_\alpha \in M,\ \lim_{\alpha\to+\infty} \frac{d(x_\alpha, \partial M)}{r_\alpha} = +\infty \text{ and } u \in \mathcal{D}_k^2(\mathbb{R}^n) \text{ satisfies } (3.2) \right\}
$$
$$
\text{or } \left\{ x_\alpha \in \partial M \text{ and } u \in \mathcal{D}_k^2(\mathbb{R}^n_-) \text{ satisfies } (3.3) \right\}
$$

If $x_\alpha \in M$, we let $\tilde{r}_\alpha > 0$ be such that

$$
\lim_{\alpha\to+\infty} \tilde{r}_\alpha = \tilde{r}_\infty \in \left[0, \frac{i_g(\tilde{M})}{2}\right),\quad \lim_{\alpha\to+\infty} \frac{r_\alpha}{\tilde{r}_\alpha} = 0 \text{ and } \tilde{r}_\alpha < \frac{d_g(x_\alpha, \partial M)}{2} \tag{3.8}
$$

and we define

$$
B_{x_\alpha, r_\alpha}(u) := \eta\left(\frac{exp_{x_\alpha}^{-1}(x)}{\tilde{r}_\alpha}\right) r_\alpha^{-\frac{n-2k}{2}} u\left(\frac{exp_{x_\alpha}^{-1}(x)}{r_\alpha}\right)
$$

where $\eta \in C_c^\infty(B_0(i_g(\tilde{M})))$ is identically 1 in a neighborhood of 0. Here, the exponential map is taken on the ambient manifold $(\tilde{M}, g)$.
If $x_\alpha \in \partial M$, we let $x_0 := \lim_{\alpha\to+\infty} x_\alpha$, and we define

$$
B_{x_\alpha, r_\alpha}(u) := \eta\left(\mathcal{T}_{x_\alpha}^{-1}(x)\right) r_\alpha^{-\frac{n-2k}{2}} u\left(\frac{\mathcal{T}_{x_\alpha}^{-1}(x)}{r_\alpha}\right)
$$

where $\mathcal{T}_x$ is as in (3.7), Ω is a neighborhood of $x_0 \in \partial M$ and $\eta \in C_c^\infty(U)$ is identically 1 in a neighborhood of 0.

Beside $[(x_\alpha), (r_\alpha), u]$, the definition of a bubble depends on the choice of the cut-off function η, the radius $\tilde{r}_\alpha$ and the chart $\mathcal{T}_x$. However, as shown in the proposition below, after quotienting by sequences going to 0, the class of a Bubble is independent of these later parameters.

Proposition 3.2.1. *The definition of Bubbles depend only on* $[(x_\alpha),(r_\alpha),u]$, *up to a sequence going to* 0 *strongly in* $H^2_{k,0}(M)$.

Proof of Proposition 3.2.1. We first assume that $u\in\mathcal{D}^2_k(\mathbb{R}^n)$ satisfies (3.2) and that

$$\lim_{\alpha\to+\infty}\frac{d_g(x_\alpha,\partial M)}{r_\alpha}=+\infty. \tag{3.9}$$

For $i=1,2$, we set the bubbles $B^i_\alpha:=\eta^i\left((\tilde r^i_\alpha)^{-1}\exp^{-1}_{x_\alpha}(\cdot)\right)r_\alpha^{-\frac{n-2k}{2}}u\left(r_\alpha^{-1}\exp^{-1}_{x_\alpha}(\cdot)\right)$, where $\eta^i\in C^\infty_c(B_0(2a_i))$, $\eta^i\equiv 1$ in $B_0(a_i)$ with $0<2a_i\le\iota_g(\tilde M)$; $\tilde r^i_\alpha>0$ are as in (3.8). We let $r^{max}_\alpha=\max\{a_1\tilde r^1_\alpha,a_2\tilde r^2_\alpha\}$ and $r^{min}_\alpha=\min\{a_1\tilde r^1_\alpha,a_2\tilde r^2_\alpha\}$, and let $\epsilon^{max}_\alpha=r_\alpha/r^{max}_\alpha$ and $\epsilon^{min}_\alpha=r_\alpha/r^{min}_\alpha$. Then $\lim_{\alpha\to0}\epsilon^{max}_\alpha=0$ and $\lim_{\alpha\to0}\epsilon^{min}_\alpha=0$. The comparison lemma 9.1 of [**13**] yields $C>0$ such that for any $R>0$ and α large

$$\begin{aligned}\sum_{l=0}^k\|\Delta_g^{l/2}\left(B^1_\alpha-B^2_\alpha\right)\|_2^2 &\le \sum_{l=0}^k\int_{B_{2r^{max}_\alpha}(x_\alpha)\setminus B_{r^{min}_\alpha}(x_\alpha)}\left(\Delta_g^{l/2}\left(B^1_\alpha-B^2_\alpha\right)\right)^2\,dv_g\\ &\le \sum_{i=1,2}\sum_{l=0}^k\int_{M\setminus B_{Rr_\alpha}(x_\alpha)}\left(\Delta_g^{l/2}\left(B^i_{x_\alpha,r_\alpha}(u)\right)\right)^2\,dv_g.\end{aligned}$$

Therefore, using (3.33), we get that $B^1_\alpha-B^2_\alpha=o(1)$ in $H^2_k(M)$ as $\alpha\to+\infty$.

Now we consider the case of a boundary bubble, that is $x_\alpha\in\partial M$ and and $u\in\mathcal{D}^2_k(\mathbb{R}^n_-)$ satisfies (3.3). For $i=1,2$, we set $B^i_\alpha:=\eta^i\left({\mathcal{T}^1}^{-1}_{x_\alpha}(\cdot)\right)r_\alpha^{-\frac{n-2k}{2}}u\left(r_\alpha^{-1}{\mathcal{T}^i}^{-1}_{x_\alpha}(\cdot)\right)$ where $\mathcal{T}_x$, $i=1,2$, are as in (3.7), U is a neighborhood of $x_0\in\partial M$ and $\eta^1,\eta^2\in C^\infty_c(U)$ are identically 1 in a neighborhood of 0. One has

$$\begin{aligned}&\sum_{l=0}^k\int_M\left(\Delta_g^{l/2}\left(B^1_\alpha-B^2_\alpha\right)\right)^2\,dv_g\le\\ &\sum_{l=0}^k\int_{D_\alpha(R)\cap M}\left(\Delta_g^{l/2}\left(B^1_\alpha-B^2_\alpha\right)\right)^2\,dv_g+\sum_{l=0}^k\int_{M\setminus D_\alpha(R)}\left(\Delta_g^{l/2}\left(B^1_\alpha-B^2_\alpha\right)\right)^2\,dv_g\end{aligned}$$

where $D_\alpha(R):=\mathcal{T}^1_{x_\alpha}(B_0(r_\alpha R))\cup\mathcal{T}^2_{x_\alpha}(B_0(r_\alpha R))$ It follows as in the comparison Lemma 9.1 of [**13**] that there exists $C>0$ such that for α large

$$\begin{aligned}&\sum_{l=0}^k\int_{D_\alpha(R)\cap M}\left(\Delta_g^{l/2}\left(B^1_\alpha-B^2_\alpha\right)\right)^2\,dv_g\le\\ &C\sum_{l=0}^k\int_{\left(B_0(r_\alpha R)\cup\Phi_\alpha^{-1}(B_0(r_\alpha R))\right)\cap\mathbb{R}^n_-}\left(\Delta^{l/2}\left((B^1_\alpha\circ\mathcal{T}^1_{x_\alpha})-(B^2_\alpha\circ\mathcal{T}^1_{x_\alpha})\right)\right)^2\,dx\le\\ &C\sum_{l=0}^k\int_{B_0(R)\cap\mathbb{R}^n_-}\left[\Delta^{l/2}\left(\eta^a(r_\alpha\cdot)u\right)-\Delta^{l/2}\left(\eta^b\left(\Phi_\alpha(r_\alpha\cdot)\right)u\left(r_\alpha^{-1}\Phi_\alpha(r_\alpha\cdot)\right)\right)\right]^2\,dx=o(1)\end{aligned}$$

where $\Phi_\alpha:={\mathcal{T}^2_{x_\alpha}}^{-1}\circ\mathcal{T}^1_{x_\alpha}$ and $d(\Phi_\alpha)_0=Id$. Similarly to the case (3.9), we get that

$$\lim_{R\to+\infty}\lim_{\alpha\to+\infty}\sum_{l=0}^k\int_{M\setminus D_\alpha(R)}\left(\Delta_g^{l/2}\left(B^1_\alpha-B^2_\alpha\right)\right)^2\,dv_g=0.$$

This completes the proof of Proposition 3.2.1. □

3.3. Preliminary analysis

The proof of Theorem 3.1 goes through four steps. All results are up to a subsequence. We let $(u_\alpha)_\alpha \in H^2_{k,0}(M)$ be a Palais-Smale sequence for I.

Step 1: We claim that $(u_\alpha)_\alpha$ is bounded in $H^2_{k,0}(M)$.

Proof of the claim: Since (u_α) is a Palais-Smale sequence, we have that

$$\begin{aligned}\langle DI(u_\alpha), u_\alpha\rangle &= \int_M (\Delta_g^{k/2} u_\alpha)^2\, dv_g + \sum_{\alpha=0}^{k-1}\int_M A_l(\nabla^l u_\alpha, \nabla^l u_\alpha)\, dv_g \\ &\quad - \int_M |u_\alpha|^{2^\sharp_k}\, dv_g = o\left(\|u_\alpha\|_{H^2_k}\right)\end{aligned}$$

Therefore

$$\int_M |u_\alpha|^{2^\sharp_k}\, dv_g = \frac{n}{k} I(u_\alpha) + o\left(\|u_\alpha\|_{H^2_k}\right) \le C + o\left(\|u_\alpha\|_{H^2_k}\right) \tag{3.10}$$

Since $(I(u_\alpha))_\alpha$ is bounded, then putting together these equalities yields

$$\|u_\alpha\|^2_{H^2_k} \le C + C\,\|u_\alpha\|^2_{H^2_{k-1}} + C\int_M |u_\alpha|^{2^\sharp_k}\, dv_g$$

Now since the embedding of $H^2_{k,0}(M)$ in $H^2_{0,k-1}(M)$ is compact, then for any $\varepsilon > 0$ there exists a $B_\varepsilon > 0$ such that $\|u\|^2_{H^2_{k-1}} \le \varepsilon\,\|u\|^2_{H^2_k} + B_\varepsilon\,\|u\|^2_{2^\sharp_k}$ for all $u \in H^2_k(M)$. Therefore, taking $\varepsilon > 0$ small enough, we get that

$$\|u_\alpha\|^2_{H^2_k} \le C + C\int_M |u_\alpha|^{2^\sharp_k}\, dv_g$$

Then using (3.10) we get that $\|u_\alpha\|^2_{H^2_k} \le C + C\,\|u_\alpha\|_{H^2_k}$ for all α, and therefore the sequence (u_α) is bounded in $H^2_{k,0}(M)$. This proves the claim. □

Since (u_α) is bounded in $H^2_{k,0}(M)$, there exists $u_\infty \in H^2_{k,0}(M)$ such that

$$\begin{cases} u_\alpha \rightharpoonup u_\infty & \text{weakly in } H^2_{k,0}(M) \text{ and } L^{2^\sharp_k}(M), \\ u_\alpha \to u_\infty & \text{strongly in } H^2_{l,0}(M) \text{ and in } L^q(M) \text{ for } l < k,\ q < 2^\sharp_k, \\ u_\alpha(x) \to u_\infty(x) & \text{a.e in } M \end{cases} \tag{3.11}$$

We define $v_\alpha := u_\alpha - u_\infty$.

Step 2: We claim that

(1) $DI(u_\infty) = 0$
(2) (v_α) is a Palais-Smale sequence for the functional J on the space $H^2_{k,0}(M)$,
(3) $J(v_\alpha) = I(u_\alpha) - I(u_\infty) + o(1)$ as $\alpha \to +\infty$.

where

$$J(u) := \frac{1}{2}\int_M (\Delta_g^{k/2} u)^2\, dv_g - \frac{1}{2^\sharp_k}\int_M |u|^{2^\sharp_k}\, dv_g \text{ for } u \in H^2_{k,0}(M)$$

Proof of the claim: We fix $\varphi \in H^2_{k,0}(M)$. We have that

$$
\begin{aligned}
\langle DI(u_\alpha),\varphi\rangle &= \int_M \Delta_g^{k/2}u_\alpha \Delta_g^{k/2}\varphi\, dv_g + \sum_{\alpha=0}^{k-1}\int_M A_l(g)(\nabla^l u_\alpha,\nabla^l\varphi)\\
&- \int_M |u_\alpha|^{2_k^\sharp-2}u_\alpha\varphi\, dv_g = o(1) \qquad (3.12)
\end{aligned}
$$

The following classical integration Lemma will be often used in the sequel (see Lemma 6.2.7 in Hebey [**10**] for a proof):

Lemma 3.3.1. *Let (M,g) be a Riemannian manifold. If (f_α) is a bounded sequence in $L^p(M)$, $1<p<+\infty$, such that $f_\alpha\to f$ a.e in M, then $f\in L^p(M)$ and $f_\alpha\rightharpoonup f$ weakly in $L^p(M)$.*

Since $(|u_\alpha|^{2_k^\sharp-2}u_\alpha)_\alpha$ is bounded in $L^{\frac{2_k^\sharp}{2_k^\sharp-1}}$ and converges a.e., Lemma 3.3.1 yields

$$
\int_M |u_\alpha|^{2_k^\sharp-2}u_\alpha\varphi\, dv_g = \int_M |u_\infty|^{2_k^\sharp-2}u_\infty\varphi\, dv_g + o(1) \qquad (3.13)
$$

Therefore, the weak convergence of (u_α) to u_∞, (3.12) and (3.13) yield that u_∞ is a weak solution to (3.1). This proves point (1) of Step 2.

We now estimate $I(u_\alpha)$. From (3.11) we have

$$
\begin{aligned}
&\int_M (\Delta_g^{k/2}u_\alpha)^2\, dv_g - \int_M (\Delta_g^{k/2}u_\infty)^2\, dv_g = \int_M (\Delta_g^{k/2}v_\alpha)^2\, dv_g + o(1),\\
&\sum_{l=0}^{k-1}\int_M A_l(\nabla^l u_\alpha,\nabla^l u_\alpha)\, dv_g = \sum_{l=0}^{k-1}\int_M A_l(\nabla^l u_\infty,\nabla^l u_\infty)\, dv_g + o(1)
\end{aligned}
$$

The following two inequalities will be of constant use in the sequel: for any $1<p<+\infty$, there exists $C>0$ such that

$$
\big|\, |a+b|^p-|a|^p-|b|^p\,\big| \le C\left(|a|^{p-1}|b|+|b|^{p-1}|a|\right) \qquad (3.14)
$$

$$
\big|\, |a+b|^p(a+b)-|a|^pa-|b|^pb\,\big| \le C\left(|a|^{p}|b|+|b|^{p}|a|\right) \qquad (3.15)
$$

for all $a,b\in\mathbb{R}$. It then follows from (3.14) that

$$
\left||u_\alpha|^{2_k^\sharp}-|u_\infty|^{2_k^\sharp}-|v_\alpha|^{2_k^\sharp}\right| \le C\left(|v_\alpha|^{2_k^\sharp-1}|u_\infty|+|u_\infty|^{2_k^\sharp-1}|v_\alpha|\right),
$$

and then using Lemma 3.3.1, we get that

$$
\int_M |u_\alpha|^{2_k^\sharp}\, dv_g - \int_M |u_\infty|^{2_k^\sharp}\, dv_g = \int_M |v_\alpha|^{2_k^\sharp}\, dv_g + o(1)
$$

Hence $I(u_\alpha)-I(u_\infty)=J(v_\alpha)+o(1)$ as $\alpha\to+\infty$, which proves point (3) of Step 2.

Next we show the sequence (v_α) is a Palais-Smale sequence for the functional J on $H_{k,0}^2(M)$. Let $\varphi\in H_{k,0}^2(M)$, we have

$$
\langle DJ(v_\alpha),\varphi\rangle = \langle DI(u_\alpha),\varphi\rangle - \langle DI(u_\infty),\varphi\rangle + \int_M \Phi_\alpha\varphi\, dv_g + o(\|\varphi\|_{H_k^2}) \qquad (3.16)
$$

where

$$
\Phi_\alpha := |v_\alpha+u_\infty|^{2_k^\sharp-2}(v_\alpha+u_\infty) - |u_\infty|^{2_k^\sharp-2}u_\infty - |v_\alpha|^{2_k^\sharp-2}v_\alpha
$$

Inequality (3.15) and Hölder's inequality yield

$$
\left|\int_M \Phi_\alpha\varphi\, dv_g\right| \le C\left(\left\||v_\alpha|^{2_k^\sharp-2}u_\infty\right\|_{\frac{2_k^\sharp}{2_k^\sharp-1}} + \left\||u_\infty|^{2_k^\sharp-2}v_\alpha\right\|_{\frac{2_k^\sharp}{2_k^\sharp-1}}\right)\|\varphi\|_{2_k^\sharp} \qquad (3.17)
$$

Since $v_\alpha \rightharpoonup 0$ in $L^{2_k^\sharp}(M)$, Lemma 3.3.1 yields

$$\left\| |v_\alpha|^{2_k^\sharp-2}u_\infty \right\|_{\frac{2_k^\sharp}{2_k^\sharp-1}} + \left\| |u_\infty|^{2_k^\sharp-2}v_\alpha \right\|_{\frac{2_k^\sharp}{2_k^\sharp-1}} = o(1)$$

Since (u_α) is a Palais-Smale for I, then (3.16), (3.17) and the continuous embedding $H_{k,0}^2(M) \hookrightarrow L^{2_k^\sharp}(M)$ yields $\langle DJ(v_\alpha), \varphi\rangle = o(\|\varphi\|_{H_k^2})$ as $\alpha \to +\infty$ uniformly wrt $\varphi \in H_{k,0}^2(M)$. This proves the claim and ends Step 2. □

The next lemma adresses the compactness of a Palais-Smale sequence for small energy. It will be generalized to the case of small local energy in Proposition 3.4.1.

Step 3: Let (v_α) be a Palais-Smale sequence for J on $H_{k,0}^2(M)$. We assume that $v_\alpha \rightharpoonup 0$ weakly in $H_{k,0}^2(M)$, and that $J(v_\alpha) \to \beta$ with $\beta < \beta^\sharp$, where $\beta^\sharp$ is as in (3.4). We claim that $v_\alpha \to 0$ strongly in $H_{k,0}^2(M)$.

Proof of the claim: Since (v_α) is bounded and $\langle DJ(v_\alpha), v_\alpha\rangle = o(\|v_\alpha\|_{H_k^2})$, we get that

$$J(v_\alpha) = \frac{k}{n}\int_M (\Delta_g^{k/2}v_\alpha)^2\, dv_g + o(1) = \frac{k}{n}\int_M |v_\alpha|^{2_k^\sharp}\, dv_g + o(1) = \beta + o(1). \tag{3.18}$$

As a consequence, $\beta \geq 0$. It follows from Mazumdar [**13**] that for any $\varepsilon > 0$ there exists $B_\varepsilon > 0$ such that

$$\|u\|_{2_k^\sharp}^2 \leq (K_0(n,k)+\varepsilon)\int_{\tilde{M}} (\Delta_g^{k/2}u)^2\, dv_g + B_\varepsilon\, \|u\|_{H_{k-1}^2}^2 \tag{3.19}$$

for all $u \in H_k^2(\tilde{M})$. Applying this inequality to v_α, the strong convergence to 0 in H_{k-1}^2 and (3.18) yield

$$\left(\frac{n}{k}\beta\right)^{2/2_k^\sharp} \leq (K_0(n,k)+\varepsilon)\,\frac{n}{k}\beta$$

Letting $\varepsilon \to 0$ and using $0 \leq \beta < \beta^\sharp$, we get that $\beta = 0$, and then (3.18) yields $v_\alpha \to 0$ strongly in $H_{k,0}^2(M)$. This proves the claim and ends Step 3. □

Step 4: Proof of Theorem 3.1. Let (u_α) be a Palais-Smale sequence for the functional I on the space $H_{k,0}^2(M)$. By substracting the weak limit u_∞, we get a Palais-Smale sequence (v_α) for the functional J with energy $J(v_\alpha) = I(u_\alpha) - I(u_\infty) + o(1)$ as $\alpha \to +\infty$. If $v_\alpha \to 0$ strongly in $H_{k,0}^2(M)$, then we end the process. If not, we apply Lemma 3.4.1 to substract a bubble modeled on $v \in \mathcal{D}_k^2(\mathbb{R}^n)\setminus\{0\}$ and we get a new Palais-Smale sequence for J, but with the energy decreased by $E(v)$. If the resulting sequence goes strongly to 0, we stop the process, if not, we iterate it again. This process must stop since the energy $E(v) \geq \beta^\sharp$ and after finitely many steps, the energy goes below the critical threshold $\beta^\sharp$ and then the convergence is strong by Step 3. This proves Theorem 3.1. □

The rest of the chapter is devoted to the proof of Lemma 3.4.1.

3.4. Extraction of a Bubble

In the sequel, for any (M,g) as in the introduction, we let $H_k^2(M)$ be the completion of $\{u \in C^\infty(M) : \|u\|_{H_k^2} < +\infty\}$ for the norm $\|\cdot\|_{H_k^2}$. The space $H_{k,0}^2(M)$ is then a closed subspace of $H_k^2(M)$. The following lemma is the main ingredient in the proof of Theorem 3.1

Lemma 3.4.1. *Let (v_α) be a Palais-Smale sequence for the functional J on $H_{k,0}^2(M)$ such that $v_\alpha \rightharpoonup 0$ weakly in $H_{k,0}^2(M)$ but not strongly. Then there exists a bubble $(B_{x_\alpha,r_\alpha}(v))$ such that upto a subsequence, the following holds:*

- *$w_\alpha := v_\alpha - B_{x_\alpha,r_\alpha}(v)$ is a Palais-Smale sequence for J,*
- *$J(w_\alpha) = J(v_\alpha) - E(v) + o(1)$ as $\alpha \to +\infty$.*

The proof of this lemma goes through 10 steps.

Step 1: We prove a strong convergence Lemma for small energies. This is a localized version of Step 3 of Section 3.3.

Proposition 3.4.1. *Let (N, g_∞) be a Riemannian manifold with positive injectivity radius.*

- *Let $(g_i)_i$ be metrics on N such that $g_i \to g_\infty$ in C^p_{loc} as $i \to +\infty$ for all p.*
- *Let $(P_i)_i$ be a family of operators on $C^\infty(N)$ such that*

$$P_i := \Delta_{g_i}^k + \sum_{l=0}^{k-1} (-1)^l \nabla^{i_1\dots i_l}\left((A_l^i)_{i_1\dots i_l j_1\dots j_l} \nabla^{j_1\dots j_l}\right)$$

with families of symmetric tensors $(A_l^i) \to A_l$ in C^p_{loc} as $i \to +\infty$ for all p.

- *We fix $\Omega \subset N$ an open smooth domain, and we define*

$$J_i(u) := \frac{1}{2}\int_\Omega u P_i u \, dv_{g_i} - \frac{1}{2_k^\sharp}\int_\Omega |u|^{2_k^\sharp} \, dv_{g_i} \text{ for } u \in H_k^2(\Omega), \tag{3.20}$$

such that J_i is C^1. Here, the background metric is g_∞.

- *We let $(u_i) \in H_{k,0}^2(\Omega)$ and $u_\infty \in H_{k,0}^2(\Omega)$ be such that $u_i \rightharpoonup u_\infty$ weakly in $H_{k,0}^2(\Omega)$ as $i \to +\infty$.*
- *We assume that there exist a compact $K \subset N$ such that*

$$\lim_{i\to+\infty} \sup_{u \in H_{k,0}^2(\Omega),\, Supp\ \varphi \subset K} \frac{\langle DJ_i(u_i), \varphi\rangle}{\|\varphi\|_{H_k^2(\Omega)}} = 0$$

- *We assume that there exists $K_\infty > 0$ and $C \geq 0$ such that*

$$\left(\int_N |u|^{2_k^\sharp} \, dv_{g_\infty}\right)^{\frac{2}{2_k^\sharp}} \leq K_\infty \int_N (\Delta_{g_\infty}^{k/2} u)^2 \, dv_{g_\infty} + C\|u\|_{H_{k-1}^2}^2 \text{ for all } u \in C_c^\infty(N). \tag{3.21}$$

We fix $x_0 \in \Omega$ and $\delta \in (0, i_{g_\infty}(N)/2)$. We assume that

$$\begin{cases} B_{x_0}(2\delta) \subset K \text{ (the ball is wrt } g_\infty), \\ \displaystyle\int_{B_{x_0}(2\delta)\cap\Omega} |u_i|^{2_k^\sharp} \, dv_{g_i} \leq \left(\frac{1}{2K_\infty}\right)^{\frac{2_k^\sharp}{2_k^\sharp - 2}} \text{ for all } i \in \mathbb{N}. \end{cases} \tag{3.22}$$

Then $u_i \to u_\infty$ strongly in $H_k^2(B_{x_0}(\delta) \cap \Omega)$.

Proof of Proposition 3.4.1*:* Up to extracting a subsequence, we assume that $u_i \to u_\infty$ strongly in $H_{k-1}^2(\omega)$ as $i \to +\infty$ for $\omega \subset \Omega$ relatively compact and $u_i(x) \to u_\infty(x)$ as $i \to +\infty$ for a.e. $x \in \Omega$. Let $\eta \in C^\infty(N)$ such that $\eta(x) = 1$ for $x \in B_{x_0}(\delta)$ and $\eta(x) = 0$ for $x \in N \setminus B_{x_0}(2\delta)$. Since η has compact support, we get that $\eta^2(u_i - u_\infty) \in H^2_{k,0}(\Omega)$ is uniformly bounded in $H^2_{k,0}(\Omega)$. Since $B_{x_0}(2\delta) \subset K$, it then follows from hypothesis (3.20) that

$$\langle DJ_i(u_i), \eta^2(u_i - u_\infty)\rangle = o(1) \text{ as } i \to +\infty.$$

Since $\eta^2(u_i - u_\infty) \to 0$ strongly in $H^2_{k-1}(\Omega)$, we then get that

$$\int_\Omega \Delta_{g_i}^{k/2} u_i \Delta_{g_i}^{k/2}(\eta^2(u_i - u_\infty))\, dv_{g_i} = \int_\Omega |u_i|^{2^\sharp_k - 2} u_i \eta^2 (u_i - u_\infty)\, dv_{g_i} + o(1) \tag{3.23}$$

as $i \to +\infty$. The weak convergence of u_i to u_∞ and the strong convergence of g_i to g_∞ on compact sets yields

$$\int_\Omega \Delta_{g_i}^{k/2} u_i \Delta_{g_i}^{k/2}(\eta^2(u_i - u_\infty))\, dv_{g_i} = \int_\Omega \Delta_{g_i}^{k/2}(u_i - u_\infty) \Delta_{g_i}^{k/2}(\eta^2(u_i - u_\infty))\, dv_{g_i} + o(1) \tag{3.24}$$

as $i \to +\infty$. As one checks, for any $\varphi \in H^2_k(\Omega)$, we have that $\Delta_{g_i}^{k/2}\varphi \Delta_{g_i}^{k/2}(\eta^2\varphi) = \left(\Delta_{g_i}^{k/2}(\eta\varphi)\right)^2 + \sum_{p<k, l\le k} \nabla^p\varphi \star \nabla^l\varphi$, where $A \star B$ denotes a linear combination of bilinear forms in A and B. Therefore, using again the strong convergence of $\eta^2(u_i - u_\infty)$ to 0 in H^2_{k-1}, we get that

$$\int_\Omega \Delta_{g_i}^{k/2} u_i \Delta_{g_i}^{k/2}(\eta^2(u_i - u_\infty))\, dv_{g_i} = \int_\Omega \left(\Delta_{g_i}^{k/2}(\eta(u_i - u_\infty))\right)^2 dv_{g_i} + o(1) \tag{3.25}$$

as $i \to +\infty$. Moreover, since $|u_i|^{2^\sharp_k - 2}\eta^2(u_i - u_\infty)$ is uniformly bounded in $L^{2^\sharp_k/(2^\sharp_k - 1)}$ and goes to 0 almost everywhere as $i \to +\infty$, then it goes weakly to 0 in $L^{2^\sharp_k/(2^\sharp_k-1)}$, and then $\int_\Omega |u_i|^{2^\sharp_k - 2}\eta^2(u_i - u_\infty) u_\infty\, dv_{g_i} \to 0$ as $i \to +\infty$. Therefore, plugging (3.24) and (3.25) into (3.23), we get that

$$\int_\Omega \left(\Delta_{g_i}^{k/2}(\eta(u_i - u_\infty))\right)^2 dv_{g_i} = \int_\Omega |u_i|^{2^\sharp_k - 2}(\eta(u_i - u_\infty))^2\, dv_{g_i} + o(1)$$

as $i \to +\infty$. Since $g_i \to g_\infty$ as $i \to +\infty$ in C^p locally on compact sets and $\eta(u_i - u_\infty)$ is uniformly bounded in $H^2_k(\Omega)$, we get that

$$\int_\Omega \left(\Delta_{g_\infty}^{k/2}(\eta(u_i - u_\infty))\right)^2 dv_{g_\infty} = \int_\Omega |u_i|^{2^\sharp_k - 2}(\eta(u_i - u_\infty))^2\, dv_{g_\infty} + o(1)$$

as $i \to +\infty$. Hölder's inequality, the Sobolev inequality (3.21), the convergence of (g_i), the strong convergence in H^2_{k-1} and (3.22) then yields

$$\begin{aligned}
&\int_\Omega \left(\Delta_{g_\infty}^{k/2}(\eta(u_i - u_\infty))\right)^2 dv_{g_\infty} \\
&\le \left(\int_{B_{x_0}(2\delta)\cap\Omega} |u_i|^{2^\sharp_k}\, dv_{g_\infty}\right)^{\frac{2^\sharp_k - 2}{2^\sharp_k}} \left(\int_N |\eta(u_i - u_\infty)|^{2^\sharp_k}\, dv_{g_\infty}\right)^{\frac{2}{2^\sharp_k}} + o(1) \\
&\le \frac{1}{2K_\infty}\left(K_\infty \int_N \left(\Delta_{g_\infty}^{k/2}(\eta(u_i - u_\infty))\right)^2 dv_{g_\infty} + C\|\eta(u_i - u_\infty)\|^2_{H^2_{k-1}}\right) + o(1) \\
&\le \left(\int_{B_{x_0}(2\delta)\cap\Omega} |u_i|^{2^\sharp_k}\, dv_{g_i}\right)^{\frac{2^\sharp_k - 2}{2^\sharp_k}} K_\infty \int_\Omega \left(\Delta_{g_\infty}^{k/2}(\eta(u_i - u_\infty))\right)^2 dv_{g_\infty} + o(1)
\end{aligned}$$

as $i \to +\infty$. Therefore, we get that $\|\Delta_{g_\infty}^{k/2}(\eta(u_i - u_\infty))\|_2 \to 0$ as $i \to +\infty$. Since $\eta(u_i - u_\infty) \to 0$ strongly in H^2_{k-1} and η has compact support, we get that

$\eta(u_i - u_\infty) \to 0$ strongly in $H_k^2(\Omega)$, and therefore $u_i \to u_\infty$ in $H_k^2(B_{x_0}(\delta)\cap\Omega)$. Note that this is up to a subsequence. Indeed, by uniqueness, the convergence holds for the initial sequence (u_i). This proves Proposition 3.4.1. □

Step 2: Since $\langle DJ(v_\alpha), v_\alpha\rangle = o(1)$, one has

$$J(v_\alpha) = \frac{k}{n}\int_M |v_\alpha|^{2_k^\sharp}\, dv_g + o(1) = \beta + o(1) \text{ as } \alpha \to +\infty$$

where $\beta := \lim_{\alpha\to+\infty} J(v_\alpha)$. By Step 3 of Section 3.3, $\beta \geq \beta^\sharp$. Therefore, since $\overline{M}$ is compact, for any $r_0 > 0$, there exists $y_0 \in \overline{M}$ and $\lambda_0 > 0$ such that

$$\int_{B_{y_0}(r_0)\cap M} |v_\alpha|^{2_k^\sharp}\, dv_g \geq \lambda_0$$

For any $r > 0$, we set

$$\mu_\alpha(r) := \max_{x\in\overline{M}} \int_{B_x(r)\cap M} |v_\alpha|^{2_k^\sharp}\, dv_g, \tag{3.26}$$

the Levy concentration function. In particular, $\mu_\alpha(r_0) \geq \lambda_0$ for all α. We fix

$$0 < \lambda < \epsilon_0 := \min\left\{\lambda_0, \frac{1}{(2K_0(n,k))^{2_k^\sharp/(2_k^\sharp-2)}}\right\}$$

where $K_0(n,k)$ is the best constant in the Euclidean Sobolev inequality (3.5). Since $\mu_\alpha(0) = 0$, there exists $(r_\alpha)_\alpha \in (0, r_0)$ and $(x_\alpha)_\alpha \in \overline{M}$ such that:

$$\lambda = \mu_\alpha(r_\alpha) = \int_{B_{x_\alpha}(r_\alpha)\cap M} |v_\alpha|^{2_k^\sharp}\, dv_g \tag{3.27}$$

Step 3: We claim that $\lim_{\alpha\to+\infty} r_\alpha = 0$.

Proof of the claim. We argue by contradiction. If (r_α) does not go to 0 up to a subsequence, we get that there exists $\delta \in (0, i_g(\tilde{M})/2)$ such that for all $x \in M$, we have that $\int_{B_x(2\delta)\cap M} |v_\alpha|^{2_k^\sharp}\, dv_g \leq \lambda$ for all α. We apply Proposition 3.4.1 with $(N, g_\infty) = (\tilde{M}, g)$, $\Omega = M$, $P_\alpha = P$, $g_\alpha = g$, $J_\alpha = J$, and the Sobolev inequality (3.19) of [**13**], and we get $v_\alpha \to 0$ as $\alpha \to +\infty$ in $H_k^2(M \cap B_x(\delta))$ for all $x \in M$. With a finite covering, we get that $v_\alpha \to 0$ as $\alpha \to +\infty$ strongly in $H_{k,0}^2(M)$, contradicting our initial hypothesis. This proves the claim and ends Step 3. □

First assume that

$$\lim_{\alpha\to+\infty} \frac{d(x_\alpha, \partial M)}{r_\alpha} = +\infty. \tag{3.28}$$

We define

$$\tilde{v}_\alpha(x) := r_\alpha^{\frac{n-2k}{2}} u_\alpha(exp_{x_\alpha}(r_\alpha x)) \text{ for } |x| < \frac{i_g(\tilde{M})}{r_\alpha} \text{ and } |x| < \frac{d(x_\alpha, \partial M)}{r_\alpha}$$

Step 4: Suppose that (3.28) holds. We claim that there exists $v \in \mathcal{D}_k^2(\mathbb{R}^n)$ such that for any $\eta \in C_c^\infty(\mathbb{R}^n)$, we have that

$$\eta\tilde{v}_\alpha \rightharpoonup \eta v \text{ weakly in } \mathcal{D}_k^2(\mathbb{R}^n) \text{ as } k \to +\infty.$$

Proof of the claim. Fix $\eta \in C_c^\infty(\mathbb{R}^n)$, and let $R_0 > 0$ be such that Supp $\eta \subset B_0(R_0)$. We define

$$\eta_\alpha(x) := \eta\left(\frac{\exp_{x_\alpha}^{-1}(x)}{r_\alpha}\right) \text{ for } x \in B_{x_\alpha}(R_0 r_\alpha), \text{ and } \eta_\alpha(x) := 0 \text{ outside.}$$

Up to a subsequence, there exists $x_0 \in \tilde{M}$ and $\tau > 0$ such that $B_{x_\alpha}(R_0 r_\alpha) \subset B_{x_0}(\tau) \subset \tilde{M}$. It then follows from the comparison Lemma 9.1 of Mazumdar [**13**] that there exists $C > 0$ such that

$$\int_{B_0(R_0 r_\alpha)} \left(\Delta^{k/2}[(\eta_\alpha v_\alpha) \circ \exp_{x_\alpha}]\right)^2 dx \le C \int_{B_{x_\alpha}(R_0 r_\alpha)} \left(\Delta_g^{k/2}(\eta_\alpha v_\alpha)\right)^2 dv_g$$

for all α. With a change of variable, rough estimates of the differential terms and Hölder's inequality, we then get

$$\int_{B_0(R_0)} \left(\Delta^{k/2}(\eta \tilde{v}_\alpha)\right)^2 dx \le C \sum_{l=0}^{k} \int_{B_{x_\alpha}(R_0 r_\alpha)} |\nabla^l u_\alpha|_g^2 |\nabla^{k-l}\eta_\alpha|_g^2 \, dv_g$$

$$\le C \sum_{l=0}^{k} \int_{B_{x_\alpha}(R_0 r_\alpha)} r_\alpha^{2(l-k)} |\nabla^l v_\alpha|_g^2 \, dv_g \le C \sum_{l=0}^{k} \|\nabla^l v_\alpha\|^2_{\frac{2n}{n-2(k-l)}} \tag{3.29}$$

It follows from Sobolev's embedding theorem that $H_{k-l}^2(M) \subset L^{\frac{2n}{n-2(k-l)}}(M)$ for all $l = 0, ..., k$ and that this embedding is continuous. Since $(v_\alpha)_\alpha$ is bounded in H_k^2, then $(\nabla^l v_\alpha)_\alpha$ is uniformly bounded in H_{k-l}^2 (with tensorial values), and then there exists $C > 0$ such that

$$\|\nabla^l v_\alpha\|_{\frac{2n}{n-2(k-l)}} \le C \|v_\alpha\|_{H_k^2} \le C' \tag{3.30}$$

for all $\alpha > 0$ and $l = 0, ..., k$. It then follows from (3.29) that $(\eta \tilde{v}_\alpha)_\alpha$ is bounded in $\mathcal{D}_k^2(\mathbb{R}^n)$. Therefore, up to a subsequence, there exists $v_\eta \in \mathcal{D}_k^2(\mathbb{R}^n)$ such that $\eta\tilde{v}_\alpha \rightharpoonup v_\eta$ weakly in $\mathcal{D}_k^2(\mathbb{R}^n)$ as $\alpha \to +\infty$. A classical diagonal argument then yields the existence $v \in H_{k,loc}^2(\mathbb{R}^n)$ such that $\eta\tilde{v}_\alpha \rightharpoonup \eta v$ weakly in $\mathcal{D}_k^2(\mathbb{R}^n)$ as $\alpha \to +\infty$. We fix $R > 0$. For any $R' > R$, a change of variables and (3.30) yields

$$\int_{B_0(R)} |\nabla^l \eta_{R'} \tilde{v}_\alpha|_{g_\alpha}^{\frac{2n}{n-2(k-l)}} \, dv_{g_\alpha} \le \int_{B_{x_\alpha}(R_0 r_\alpha)} |\nabla^l v_\alpha|_g^{\frac{2n}{n-2(k-l)}} \, dv_g \le C$$

where $g_\alpha := \exp_{x_\alpha}^\star g(r_\alpha \cdot)$. Using weak convergence and convexity, letting $\alpha \to +\infty$ and then $R \to +\infty$ yields $|\nabla^l v| \in L^{\frac{2n}{n-2(k-l)}}(\mathbb{R}^n)$. As one checks, we then have that the sequence $(\eta_R v)_R$ is a Cauchy sequence in $\mathcal{D}_k^2(\mathbb{R}^n)$, and then we get that $v \in \mathcal{D}_k^2(\mathbb{R}^n)$. This ends the proof of the claim, and ends Step 4. □

Step 5: We assume that (3.28) holds. We let $v \in \mathcal{D}_k^2(\mathbb{R}^n)$ as in Claim 3. We claim that $v \not\equiv 0$ is a weak solution to $\Delta^k v = |v|^{2_k^\sharp - 2} v$ in $\mathcal{D}_k^2(\mathbb{R}^n)$.

Proof of the claim. We fix $R > 0$ and we apply Proposition 3.4.1 with $(N, g_\infty) := (\mathbb{R}^n, \text{Eucl})$ and $\Omega := \mathbb{R}^n$. As above, we define a family of smooth metrics $(g_\alpha)_\alpha$ such that $g_\alpha(x) := \exp_{x_\alpha}^\star g(r_\alpha x)$ for $x \in B_0(3R)$, $g_\alpha(x) = \text{Eucl}$ for $x \in \mathbb{R}^n \setminus B_0(4R)$, and $g_\alpha \to \text{Eucl}$ in $C_{loc}^p(\mathbb{R}^n)$ as $\alpha \to +\infty$ for all p. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be such that Supp $\varphi \subset B_0(R)$. We define

$$\varphi_\alpha(x) := r_\alpha^{-\frac{n-2k}{2}} \varphi\left(\frac{\exp_{x_\alpha}^{-1}(x)}{r_\alpha}\right)$$

for all $x \in M$. As one checks, φ_α is well-defined and has support in $B_{x_\alpha}(Rr_\alpha)$. Moreover, using the comparison Lemma 9.1 in Mazumdar [**13**] and arguing as in Step 4, we get that $\|\varphi_\alpha\|_{H^2_{k,0}(M)} \le C(R)\|\varphi\|_{H^2_{k,0}(\mathbb{R}^n)}$ for all $\alpha > 0$. Since (u_α) is a Palais-Smale sequence, we have that

$$\langle DJ(v_\alpha), \varphi_\alpha\rangle = o(\|\varphi_\alpha\|_{H^2_{k,0}}) = o(\|\varphi\|_{H^2_{k,0}(\mathbb{R}^n)})$$

as $\alpha \to +\infty$ uniformly for all $\varphi \in C^\infty_c(\mathbb{R}^n)$ such that Supp $\varphi \subset B_0(R)$. With a change of variable, we get $\langle DJ(v_\alpha), \varphi_\alpha\rangle = \langle DJ_\alpha(\eta_R\tilde{v}_\alpha), \varphi\rangle$ where

$$J_\alpha(u) := \frac{1}{2}\int_{\mathbb{R}^n}(\Delta_{g_\alpha}^{k/2}u)^2\,dv_{g_\alpha} - \frac{1}{2^\sharp_k}\int_{\mathbb{R}^n}|u|^{2^\sharp_k}\,dv_{g_\alpha}$$

for all $u \in H^2_k(\mathbb{R}^n)$. Therefore, $\langle DJ_\alpha(\eta_R\tilde{v}_\alpha), \varphi\rangle = o(\|\varphi\|_{H^2_{k,0}(\mathbb{R}^n)})$ as $\alpha \to +\infty$ uniformly for all $\varphi \in C^\infty_c(\mathbb{R}^n)$ such that Supp $\varphi \subset B_R(0)$.

We fix $x_0 \in \mathbb{R}^n$ such that $B_{x_0}(1/2) \subset B_0(R)$. A change of variable yields

$$\int_{B_{x_0}(1/2)\cap B_0(2R)}|\eta_R\tilde{v}_\alpha|^{2^\sharp_k}\,dv_{g_\alpha} = \int_{\exp_{x_\alpha}(r_\alpha B_{x_0}(1/2))}|u_\alpha|^{2^\sharp_k}\,dv_g.$$

For $\alpha > 0$ large enough, we have that $\exp_{x_\alpha}(r_\alpha B_{x_0}(1/2)) \subset B_{\exp_{x_\alpha}(x_0)}(r_\alpha)$. Therefore, it follows from the definition of μ_α that

$$\int_{B_{x_0}(1/2)\cap B_0(2R)}|\eta_R\tilde{v}_\alpha|^{2^\sharp_k}\,dv_{g_\alpha} \le \mu_\alpha(r_\alpha) = \lambda < \epsilon_0$$

for all α large enough and $x_0 \in \mathbb{R}^n$ such that $1/2 + |x_0| < R$. With the Sobolev inequality (3.5) on $\mathbb{R}^n$, we apply Proposition 3.4.1 to $(\eta_R\tilde{v}_\alpha)_\alpha$, and we get that

$$\lim_{\alpha\to+\infty}\eta_R\tilde{v}_\alpha = \eta_R v \text{ strongly in } H^2_k(B_{x_0}(1/4)).$$

Using a finite covering, we then have $\tilde{v}_\alpha \to v$ strongly in $H^2_k(B_0(R/2))$ as $\alpha \to +\infty$. Sobolev's embedding theorem yield the convergence in $L^{2^\sharp_k}(B_0(1))$. Since

$$\int_{B_0(1)}|\tilde{v}_\alpha|^{2^\sharp_k}\,dv_{g_\alpha} = \int_{B_{x_\alpha}(r_\alpha)}|v_\alpha|^{2^\sharp_k}\,dv_g = \mu_\alpha(r_\alpha) = \lambda > 0,$$

passing to the limit $\alpha \to +\infty$ yields $\int_{B_0(1)}|v|^{2^\sharp_k}\,dx = \lambda \neq 0$, and therefore $v \not\equiv 0$. This proves the claim and ends Step 5. □

Note that indeed, we have proved that

$$\lim_{\alpha\to+\infty}\tilde{v}_\alpha = v \text{ strongly in } H^2_k(B_0(R)) \text{ for all } R > 0. \tag{3.31}$$

We choose a sequence $(\tilde{r}_\alpha)$ of positive real numbers as in (3.8) with $\eta \in C^\infty_c(B_0(\delta))$ (with $\delta \in (0, i_g(\tilde{M}))$) identically 1 around 0. As in Definition 3.2.1, we set

$$V_\alpha(x) := B_{x_\alpha, r_\alpha}(v) := \eta\left(\frac{exp^{-1}_{x_\alpha}(x)}{\tilde{r}_\alpha}\right)\, r_\alpha^{-\frac{n-2k}{2}}\, v\left(\frac{exp^{-1}_{x_\alpha}(x)}{r_\alpha}\right)$$

We have that $V_\alpha \in H^2_{k,0}(M)$.

Step 6: We claim that

$$V_\alpha \rightharpoonup 0 \quad \text{in } H^2_{k,0}(M) \text{ as } \alpha \to +\infty. \tag{3.32}$$

Proof of the claim. We argue essentially as in [**13**]. We fix $0 \leq l \leq k$ and we define $\epsilon_\alpha := r_\alpha/\tilde{r}_\alpha$ such that $\lim_{\alpha\to+\infty} \epsilon_\alpha = 0$. We fix $R \geq 0$ (potentially 0). It follows from the comparison Lemma 9.1 of [**13**] that there exists $C > 0$ such that

$$\begin{aligned}
\int_{M\setminus B_{x_\alpha}(Rr_\alpha)} (\Delta_g^{l/2} V_\alpha)^2\, dv_g &\leq C \int_{B_0(\delta\tilde{r}_\alpha)\setminus B_0(Rr_\alpha)} (\Delta^{l/2}(V_\alpha \circ \exp_{x_\alpha}))^2\, dx \\
&\leq C r_\alpha^{2(k-l)} \int_{B_0(\delta\epsilon_\alpha^{-1})\setminus B_0(R)} \left(\Delta^{l/2}\left(\eta\left(\epsilon_\alpha \cdot\right) v\right)\right)^2 dx \\
&\leq C r_\alpha^{2(k-l)} \int_{B_0(\delta\epsilon_\alpha^{-1})\setminus B_0(R)} |\nabla^l(\eta\left(\epsilon_\alpha \cdot\right) v)|^2\, dx \\
&\leq C r_\alpha^{2(k-l)} \sum_{i=0}^{l} \int_{\mathbb{R}^n\setminus B_0(R)} |\nabla^{l-i}\eta\left(\epsilon_\alpha \cdot\right)||\nabla^i v|^2\, dx \\
&\leq C r_\alpha^{2(k-l)} \sum_{i=0}^{l} \int_{\mathbb{R}^n\setminus B_0(R)} \epsilon_\alpha^{2(l-i)} |\nabla^i v|^2\, dx
\end{aligned}$$

Since $v \in \mathcal{D}_k^2(\mathbb{R}^n)$, we have that $\nabla^i v \in \mathcal{D}_{k-i}^2(\mathbb{R}^n)$, and therefore $|\nabla^i v| \in L^{2^\sharp_{(k-i)}}(\mathbb{R}^n)$ where $2^\sharp_{(k-i)} := \frac{2n}{n-2(k-i)}$. Therefore, Hölder's inequality yields

$$\int_{M\setminus B_{x_\alpha}(Rr_\alpha)} (\Delta_g^{l/2} V_\alpha)^2\, dv_g \leq C \tilde{r}_\alpha^{2(k-l)} \sum_{i=0}^{l} \left(\int_{\mathbb{R}^n\setminus B_0(R)} |\nabla^i v|^{2^\sharp_{(k-i)}}\, dx\right)^{\frac{2}{2^\sharp_{(k-i)}}} \tag{3.33}$$

Taking $R = 0$ and $l = 0, ..., k$ yields the boundedness of $(V_\alpha)_\alpha$ in $H_{k,0}^2(M)$.

Arguing as in above, we get that for any $R > 0$ and any $l = 0, ..., k$, we have that

$$\int_{B_{x_\alpha}(Rr_\alpha)} (\Delta_g^{l/2} V_\alpha)^2\, dv_g \leq C r_\alpha^{2(k-l)} \sum_{i=0}^{l} \int_{B_0(R)} \epsilon_\alpha^{2(l-i)} |\nabla^i v|^2\, dx \tag{3.34}$$

Since $\nabla^i v \in L^2_{loc}(\mathbb{R}^n)$ for all $i = 0, ..., k$, then taking $l = 0$ in (3.33) and (3.34), letting $\alpha \to +\infty$ and then $R \to +\infty$ yields $V_\alpha \to 0$ in $L^2(M)$. Then the weak compactness of bounded sequences yields (3.32). This proves the claim and ends Step 6. □

Step 7: We claim that

$$DJ(V_\alpha) \longrightarrow 0 \text{ strongly as } \alpha \to +\infty \tag{3.35}$$

Proof of the claim. We set $\varphi \in C_c^\infty(M)$. We have that

$$\langle DJ(V_\alpha), \varphi\rangle = \int_M \Delta_g^{k/2} V_\alpha \Delta_g^{k/2}\varphi\, dv_g - \int_M |V_\alpha|^{2_k^\sharp - 2} V_\alpha \varphi\, dv_g$$

We fix $R > 0$ and we define

$$I_{R,\alpha}(\varphi) := \int_{B_{x_\alpha}(Rr_\alpha)} \Delta_g^{k/2} V_\alpha \Delta_g^{k/2}\varphi\, dv_g - \int_{B_{x_\alpha}(Rr_\alpha)} |V_\alpha|^{2_k^\sharp - 2} V_\alpha \varphi\, dv_g$$

and

$$II_{R,\alpha}(\varphi) := \int_{M\setminus B_{x_\alpha}(Rr_\alpha)} \Delta_g^{k/2} V_\alpha \Delta_g^{k/2}\varphi\, dv_g - \int_{M\setminus B_{x_\alpha}(Rr_\alpha)} |V_\alpha|^{2_k^\sharp - 2} V_\alpha \varphi\, dv_g.$$

Step 7.1: we estimate $II_{R,\alpha}(\varphi)$. Via Hölder's and Sobolev inequality, we have that

$$(3.36)\qquad |II_{R,\alpha}(\varphi)| \leq \left(\int_{D_\alpha(R)} (\Delta_g^{k/2} V_\alpha)^2 \, dv_g\right)^{\frac{1}{2}} \times \|\Delta_g^{k/2}\varphi\|_2 + \left(\int_{D_\alpha(R)} |V_\alpha|^{2_k^\sharp} \, dv_g\right)^{\frac{2_k^\sharp - 1}{2_k^\sharp}} \times \|\varphi\|_{2_k^\sharp}$$
$$\leq \left(\left(\int_{D_\alpha(R)} (\Delta_g^{k/2} V_\alpha)^2 \, dv_g\right)^{\frac{1}{2}} + \left(\int_{D_\alpha(R)} |V_\alpha|^{2_k^\sharp} \, dv_g\right)^{\frac{2_k^\sharp - 1}{2_k^\sharp}}\right) \cdot \|\varphi\|_{H_k^2}$$

with $D_\alpha(R) := M \setminus B_{x_\alpha}(Rr_\alpha)$. Lemma 9.1 in [**13**] and $v \in L^{2_k^\sharp}(\mathbb{R}^n)$ yield

$$(3.37)\qquad \int_{M \setminus B_{x_\alpha}(Rr_\alpha)} |V_\alpha|^{2_k^\sharp} \, dv_g \leq C \int_{\mathbb{R}^n \setminus B_0(Rr_\alpha)} |V_\alpha \circ \exp_{x_\alpha}|^{2_k^\sharp} \, dx \leq C \int_{\mathbb{R}^n \setminus B_0(R)} |v|^{2_k^\sharp} \, dx$$

Plugging (3.33) with $l = k$ and (3.37) into (3.36), letting $R \to +\infty$ and $\alpha \to +\infty$ yields

$$(3.38)\qquad \lim_{R \to +\infty} \lim_{\alpha \to +\infty} \frac{II_{R,\alpha}(\varphi)}{\|\varphi\|_{H_k^2}} = 0 \text{ uniformly wrt } \varphi \in H_{k,0}^2(M) \setminus \{0\}$$

Step 7.2: We now estimate $I_{R,\alpha}(\varphi)$. We define

$$\overline{\varphi}_\alpha(x) = \eta(\epsilon_\alpha x) r_\alpha^{\frac{n-2k}{2}} \varphi\left(exp_{x_\alpha}(r_\alpha x)\right)$$

where $\epsilon_\alpha := r_\alpha / \tilde{r}_\alpha$. As one checks, $\overline{\varphi}_\alpha \in C_c^\infty(\mathbb{R}^n)$. Using the comparison Lemma 9.1 in [**13**] and arguing as in (3.33)-(3.34), we get that

$$\|\overline{\varphi}_\alpha\|_{\mathcal{D}_k^2(\mathbb{R}^n)} \leq C \|\varphi\|_{H_k^2}$$

where $C > 0$ is independent of φ. As one checks,

$$I_{R,\alpha}(\varphi) = \int_{B_0(R)} \Delta_{g_\alpha}^{k/2} v \Delta_{g_\alpha}^{k/2} \overline{\varphi}_\alpha \, dv_{g_\alpha} - \int_{B_0(R)} |v|^{2_k^\sharp - 2} v \overline{\varphi}_\alpha \, dv_{g_\alpha}$$

Since $g_\alpha \to \text{Eucl}$ as $\alpha \to +\infty$ in $C_{loc}^p(\mathbb{R}^n)$ for all $p \geq 1$, we get

$$(3.39)\quad I_{R,\alpha}(\varphi) = \int_{B_0(R)} \Delta^{k/2} v \Delta^{k/2} \overline{\varphi}_\alpha \, dx - \int_{B_0(R)} |v|^{2_k^\sharp - 2} v \overline{\varphi}_\alpha \, dx + o\left(\|\overline{\varphi}_\alpha\|_{\mathcal{D}_k^2(\mathbb{R}^n)}\right)$$

where the convergence is uniform wrt $\overline{\varphi}_\alpha$. Since v is a weak solution to (3.1), then (3.39) yields

$$(3.40)\qquad \lim_{R \to +\infty} \lim_{\alpha \to +\infty} \frac{I_{R,\alpha}(\varphi)}{\|\varphi\|_{H_k^2}} = 0 \text{ uniformly wrt } \varphi \in H_{k,0}^2(M) \setminus \{0\}$$

The limits (3.38) and (3.40) yield $\langle DJ(V_\alpha), \varphi \rangle = o(\|\varphi\|_{H_k^2})$ as $\alpha \to +\infty$ uniformly wrt $\varphi \in C_c^\infty(M)$. The boundedness of (V_α) in $H_{k,0}^2(M)$ then yields $DJ(V_\alpha) \to 0$ strongly in $(H_{k,0}^2(M))'$ as $\alpha \to +\infty$. This proves (3.35) and ends Step 7. $\square$

We define $w_\alpha := v_\alpha - V_\alpha$. It follows from (3.32) that $w_\alpha \rightharpoonup 0$ weakly in $H_{k,0}^2(M)$.

Step 8: We claim that

$$DJ(w_\alpha) \longrightarrow 0 \text{ strongly} \tag{3.41}$$

Proof of the claim. For $\varphi \in H^2_{k,0}(M)$, we write

$$\langle DJ(w_\alpha), \varphi\rangle = \langle DJ(v_\alpha), \varphi\rangle - \langle DJ(V_\alpha), \varphi\rangle - \int_M \Phi_\alpha \varphi \, dv_g \tag{3.42}$$

where $\Phi_\alpha := |w_\alpha|^{2^\sharp_k - 2} w_\alpha - |v_\alpha|^{2^\sharp_k - 2} v_\alpha + |V_\alpha|^{2^\sharp_k - 2} V_\alpha$. Then by applying the Hölder and Sobolev inequalities we get

$$\left| \int_M \Phi_\alpha \varphi \, dv_g \right| \leq C \|\varphi\|_{H^2_k} \|\Phi_\alpha\|_{2^\sharp_k/(2^\sharp_k - 1)}$$

Step 8.1: We fix $R > 0$. Inequality (3.15) and Hölder's inequality yield

$$\begin{aligned}
&\int_{M\setminus B_{x_\alpha}(Rr_\alpha)} |\Phi_\alpha|^{2^\sharp_k/(2^\sharp_k-1)} \, dv_g \\
&\leq C \int_{M\setminus B_{x_\alpha}(Rr_\alpha)} \left(|v_\alpha|^{2^\sharp_k - 2} |V_\alpha| + |V_\alpha|^{2^\sharp_k - 2} |v_\alpha| \right)^{2^\sharp_k/(2^\sharp_k-1)} \, dv_g \\
&\leq C \left(\int_M |v_\alpha|^{2^\sharp_k} \, dv_g \right)^{\frac{2^\sharp_k - 2}{2^\sharp_k - 1}} \left(\int_{M\setminus B_{x_\alpha}(Rr_\alpha)} |V_\alpha|^{2^\sharp_k} \, dv_g \right)^{\frac{1}{2^\sharp_k - 1}} \\
&+ C \left(\int_M |v_\alpha|^{2^\sharp_k} \, dv_g \right)^{\frac{1}{2^\sharp_k - 1}} \left(\int_{M\setminus B_{x_\alpha}(Rr_\alpha)} |V_\alpha|^{2^\sharp_k} \, dv_g \right)^{\frac{2^\sharp_k - 2}{2^\sharp_k - 1}}
\end{aligned}$$

Since (v_α) is uniformly bounded in $H^2_k(M)$, then (3.37) yields

$$\lim_{R\to+\infty} \lim_{\alpha\to+\infty} \int_{M\setminus B_{x_\alpha}(Rr_\alpha)} |\Phi_\alpha|^{2^\sharp_k/(2^\sharp_k-1)} \, dv_g = 0. \tag{3.43}$$

This ends Step 8.1.

Step 8.2: We fix $R > 0$. A change of variable and inequality (3.15) yield

$$\begin{aligned}
&\int_{B_{x_\alpha}(Rr_\alpha)} |\Phi_\alpha|^{2^\sharp_k/(2^\sharp_k-1)} \, dv_g \\
&= \int_{B_0(R)} \left| |\tilde v_\alpha - v|^{2^\sharp_k - 2} (\tilde v_\alpha - v) - |\tilde v_\alpha|^{2^\sharp_k - 2} \tilde v_\alpha + |v|^{2^\sharp_k - 2} v \right|^{2^\sharp_k/(2^\sharp_k-1)} \, dv_{g_\alpha} \\
&\leq C \int_{B_0(R)} \left(|\tilde v_\alpha - v|^{\frac{(2^\sharp_k - 2)2^\sharp_k}{2^\sharp_k - 1}} |v|^{\frac{2^\sharp_k}{2^\sharp_k - 1}} + |v|^{\frac{(2^\sharp_k - 2)2^\sharp_k}{2^\sharp_k - 1}} |\tilde v_\alpha - v|^{\frac{2^\sharp_k}{2^\sharp_k - 1}} \right) dx
\end{aligned}$$

For any $\eta \in C^\infty_c(\mathbb{R}^n)$, we have that $\eta \tilde v_\alpha \rightharpoonup \eta v$ weakly in $\mathcal{D}^2_k(\mathbb{R}^n)$. Therefore, up to extracting a subsequence, $(\tilde v_\alpha)_\alpha$ is uniformly bounded in $L^{2^\sharp_k}(B_0(R))$ and goes to v almost everywhere as $\alpha \to +\infty$. Therefore Lemma 3.3.1 yields that for any $R > 0$,

$$\lim_{\alpha\to+\infty} \int_{B_{x_\alpha}(Rr_\alpha)} |\Phi_\alpha|^{2^\sharp_k/(2^\sharp_k-1)} \, dv_g = 0. \tag{3.44}$$

The limits (3.43)-(3.44) yield $\|\Phi_\alpha\|_{2^\sharp_k/(2^\sharp_k-1)} \to 0$ as $\alpha \to +\infty$. Then by (3.42) we get $DJ(w_\alpha) \to 0$ in $(H^2_{k,0}(M))'$ as $\alpha \to +\infty$. This proves (3.41) and ends Step 8. $\square$

Step 9: We claim that we have the following decomposition of energy.

$$J(w_\alpha) = J(v_\alpha) - E(v) + o(1) \text{ where } o(1) \to 0 \text{ as } \alpha \to +\infty. \tag{3.45}$$

Proof of the claim. As one checks,

$$J(v_\alpha) - J(w_\alpha) - J(V_\alpha) = \langle DJ(w_\alpha), V_\alpha\rangle$$
$$-\frac{1}{2^\sharp_k}\int_M \left(|w_\alpha + V_\alpha|^{2^\sharp_k} - |w_\alpha|^{2^\sharp_k} - 2^\sharp_k |w_\alpha|^{2^\sharp_k-2} w_\alpha V_\alpha - |V_\alpha|^{2^\sharp_k}\right) dv_g$$

We fix $R > 0$. Arguing as in the proof of (3.44), we get that

$$\lim_{\alpha\to+\infty}\int_{B_{x_\alpha}(Rr_\alpha)} \left(|w_\alpha + V_\alpha|^{2^\sharp_k} - |w_\alpha|^{2^\sharp_k} - 2^\sharp_k |w_\alpha|^{2^\sharp_k-2} w_\alpha V_\alpha - |V_\alpha|^{2^\sharp_k}\right) dv_g = 0.$$

As one checks, there exists $C > 0$ such that

$$\left||a+b|^{2^\sharp_k} - |a|^{2^\sharp_k} - 2^\sharp_k |a|^{2^\sharp_k-2}ab - |b|^{2^\sharp_k}\right| \le C\left(|a|^{2^\sharp_k-2}|b|^2 + |a|\cdot|b|^{2^\sharp_k-1}\right)$$

for all $a, b \in \mathbb{R}$. As in the proof of (3.43), we get that

$$\lim_{R\to+\infty}\lim_{\alpha\to+\infty}\int_{D_\alpha(R)} \left(|w_\alpha + V_\alpha|^{2^\sharp_k} - |w_\alpha|^{2^\sharp_k} - 2^\sharp_k |w_\alpha|^{2^\sharp_k-2} w_\alpha V_\alpha - |V_\alpha|^{2^\sharp_k}\right) dv_g = 0,$$

where $D_\alpha(R) := M \setminus B_{x_\alpha}(Rr_\alpha)$. These yield $J(v_\alpha) = J(w_\alpha) + J(V_\alpha) + o(1)$.

We now estimate $J(V_\alpha)$. The estimates (3.33) and (3.37) yield

$$\lim_{R\to+\infty}\lim_{\alpha\to+\infty}\int_{M\setminus B_{x_\alpha}(Rr_\alpha)} \left((\Delta_g^{k/2}V_\alpha)^2 + |V_\alpha|^{2^\sharp_k}\right) dv_g = 0$$

For $R > 0$, we have that

$$\int_{B_{x_\alpha}(Rr_\alpha)} \left(\frac{(\Delta_g^{k/2}V_\alpha)^2}{2} - \frac{|V_\alpha|^{2^\sharp_k}}{2^\sharp_k}\right) dv_g = \int_{B_0(R)} \left(\frac{(\Delta_{g_\alpha}^{k/2}v)^2}{2} - \frac{|v|^{2^\sharp_k}}{2^\sharp_k}\right) dv_{g_\alpha}$$

Since $g_\alpha \to \text{Eucl}$ locally uniformly in C^p for all p and $v \in \mathcal{D}^2_k(\mathbb{R}^n)$, we get that

$$\lim_{R\to+\infty}\lim_{\alpha\to+\infty}\int_{B_{x_\alpha}(Rr_\alpha)} \left(\frac{(\Delta_g^{k/2}V_\alpha)^2}{2} - \frac{|V_\alpha|^{2^\sharp_k}}{2^\sharp_k}\right) dv_g = \int_{\mathbb{R}^n} \left(\frac{(\Delta^{k/2}v)^2}{2} - \frac{|v|^{2^\sharp_k}}{2^\sharp_k}\right) dx$$

All these estimates yield (3.45). This ends Step 9. $\square$

Step 10: Next we deal with the case

$$d_g(x_\alpha, \partial M) = O(r_\alpha) \text{ as } \alpha \to +\infty$$

Since $r_\alpha \to 0$ as $\alpha \to +\infty$, then there exists $x_\infty \in \partial M$ such that $x_\alpha \to x_\infty$ as $\alpha \to +\infty$. For any $\alpha \in \mathbb{N}$, we let $z_\alpha \in \partial M$ be such that

$$d_g(x_\alpha, z_\alpha) = d_g(x_\alpha, \partial M)$$

In particular, $\lim_{\alpha\to+\infty} z_\alpha = x_\infty$. We choose a family of charts $z \mapsto \mathcal{T}_z$ for $z \in \Omega\cap\partial M$ as in (3.7). Since the $d(\mathcal{T}_z)_0$ is an isometry, there exists $\mathcal{C}_1, \mathcal{C}_2 > 0$, $\tau_1, \tau_2 > 0$ such that for any $z \in \Omega \cap \partial M$, $r < \tau_1$ and $y \in \mathbb{R}^n_- \cap B_0(\tau_2)$, one has

$$B_{\mathcal{T}_z(y)}(\mathcal{C}_1 r) \cap M \subset \mathcal{T}_z\left(B_y(r) \cap \mathbb{R}^n_-\right) \subset B_{\mathcal{T}_z(y)}(\mathcal{C}_2 r) \cap M$$

For $x \in r_\alpha^{-1}U \cap \{x_1 < 0\}$, we define

$$\tilde{v}_\alpha(x) := r_\alpha^{\frac{n-2k}{2}} v_\alpha \circ \mathcal{T}_{z_\alpha}(r_\alpha x) \text{ and } \tilde{g}_\alpha(x) := \mathcal{T}_{z_\alpha}{}^\star g\ (r_\alpha x)$$

As one checks, for any $\eta \in C_c^\infty(\mathbb{R}^n)$, we have that $\eta\tilde{v}_\alpha \in \mathcal{D}_k^2(\mathbb{R}^n_-)$. Arguing as Step 4, we get that there exists $v \in \mathcal{D}_k^2(\mathbb{R}^n_-)$ such that

$$\eta\tilde{v}_\alpha \rightharpoonup \eta v \text{ weakly in } \mathcal{D}_k^2(\mathbb{R}^n_-) \text{ as } \alpha \to +\infty.$$

Moreover, using Proposition 3.4.1 and arguing as in Step 5, we get that $v \not\equiv 0$ is a weak solution to (3.3) and $\tilde{v}_\alpha \to v$ as $\alpha \to +\infty$ strongly in $H_k^2(B_0(R) \cap \mathbb{R}^n_-)$ for all $R > 0$. As in Definition 3.2.1, for $\alpha \in \mathbb{N}$ and $x \in \overline{M}$, we set

$$V_\alpha(x) := B_{z_\alpha, r_\alpha}(v)(x) = \eta\left(\mathcal{T}_{z_\alpha}^{-1}(x)\right) r_\alpha^{-\frac{n-2k}{2}} v\left(r_\alpha^{-1}\mathcal{T}_{z_\alpha}{}^{-1}(x)\right)$$

We define $w_\alpha := v_\alpha - V_\alpha$. Arguing as in Steps 6 to 9, we get that

- $w_\alpha \rightharpoonup 0$ weakly in $H_{k,0}^2(M)$
- $DJ(w_\alpha) \to 0$ weakly in $(H_{k,0}^2(M))'$
- $J(w_\alpha) = J(v_\alpha) - E(v) + o(1)$

as $\alpha \to +\infty$. This completes the proof of Lemma 3.4.1.

3.5. Nonnegative Palais-Smale sequences

To prove Theorem 3.2, we first set the following property:

Proposition 3.5.1. *Let (u_α) be a Palais-Smale sequence for the functional I on the space $H_{k,0}^2(M)$. Let $d \in \mathbb{N}$ and let $[(x_\alpha^{(j)}), (r_\alpha^{(j)}), u^{(j)}]$, $j = 1, ..., d$, be d bubbles as in Theorem 3.1. Then, for any $N \in \{1, \dots, d\}$, there exists $L \geq 0$ sequences $(y_\alpha^j)_{\alpha>0} \in \overline{M}$ and $(\lambda_\alpha^j)_{\alpha>0} \in (0, +\infty)$, $j = 1, \cdots, L$, such that for any $R > 0$*

$$\lim_{R'\to+\infty}\lim_{\alpha\to+\infty}\int_{\left(B_{x_\alpha^N}(Rr_\alpha^N)\setminus\bigcup_{j=1}^L B_{y_\alpha^j}(R'\lambda_\alpha^j)\right)\cap M} |u_\alpha - B_{x_\alpha^{(N)}, r_\alpha^{(N)}}(u^{(N)})|^{2_k^\sharp}\, dv_g = 0$$

where for any j, $j = 1, \cdots, L$, $d_g(x_\alpha^N, y_\alpha^j) = o(r_\alpha^N)$ and $\lambda_\alpha^j = o(r_\alpha^N)$ as $\alpha \to +\infty$. Moreover, we have that

$$\lim_{\alpha\to+\infty} \frac{d_g(x_\alpha^i, x_\alpha^j)^2}{r_\alpha^i r_\alpha^j} + \frac{r_\alpha^i}{r_\alpha^j} + \frac{r_\alpha^j}{r_\alpha^i} = +\infty \textit{ for all } i \neq j \in \{1, ..., d\}.$$

We omit the proof which goes exactly as in Hebey-Robert [**11**]. Here we use the boundary chart (3.7) for bubbles accumulating on the boundary.

We now prove Theorem 3.2. We let $(u_\alpha)_\alpha$ be as in the statement of the theorem, and we let $[(x_\alpha^{(j)}), (r_\alpha^{(j)}), u^{(j)}]$, $j = 1, ..., d$, be the associated bubbles. We fix $N \in \{1, ..., d\}$. For simplicity, we define $r_\alpha := r_\alpha^{(N)}$ and $x_\alpha := x_\alpha^{(N)}$. We assume that $r_\alpha^{-1}d(x_\alpha, \partial M) \to +\infty$ as $\alpha \to +\infty$. It then follows from Proposition 3.5.1 that there exists a finite set $\mathcal{S} \subset \mathbb{R}^n$ such that $\lim_{\alpha\to+\infty} \tilde{v}_\alpha = u^N$ strongly in $L_{loc}^{2_k^\sharp}(\mathbb{R}^n \setminus \mathcal{S})$ where $\tilde{v}_\alpha(x) := r_\alpha^{\frac{n-2k}{2}} u_\alpha(\exp_{x_\alpha}(r_\alpha x))$ for $x \in \mathbb{R}^n$. Up to extracting a subsequence, the convergence holds a.e. Since $u_\alpha \geq 0$, we then get that $u^N \geq 0$. It then follows from Lemma 4 in Ge-Wei-Zhou [**7**] that there exists $\lambda > 0$ and $a \in \mathbb{R}^n$ such that $u^N = U_{\lambda,a}$ is of the form (3.6).

We claim that $u^N = U_{\lambda,0}$, that is $a = 0$. We prove the claim. Indeed, rescaling (3.26) and (3.27) yields

$$\int_{r_\alpha^{-1}\exp_{x_\alpha}^{-1}(B_{\exp_{x_\alpha}(r_\alpha x)}(r_\alpha))} |\tilde{v}_\alpha|^{2_k^\sharp}\, dv_{g_\alpha} \leq \int_{B_0(1)} |\tilde{v}_\alpha|^{2_k^\sharp}\, dv_{g_\alpha}$$

for all $z \in \mathbb{R}^n$ and α large enough. Since the exponential map is a normal chart and is an isometry at x_α, we get that for all $z \in \mathbb{R}^n$ and all $\epsilon > 0$

$$\exp_{x_\alpha}(r_\alpha B_z(1-\epsilon)) \subset B_{\exp_{x_\alpha}(r_\alpha z)}(r_\alpha).$$

Plugging these two inequalities together and letting $\alpha \to +\infty$, using the strong convergence (3.31), we get that $\int_{B_z(1-\epsilon)} |u^N|^{2_k^\sharp}\, dx \leq \int_{B_0(1)} |u^N|^{2_k^\sharp}\, dx$. Letting $\epsilon \to 0$ yields

$$\int_{B_z(1)} |u^N|^{2_k^\sharp}\, dx \leq \int_{B_0(1)} |u^N|^{2_k^\sharp}\, dx.$$

As one checks, since $u^N = U_{\lambda,a}$, where $U_{\lambda,a}$ is as in (3.6), the maximum of the left-hand-side is achieved if and only if $z = a$. Therefore $a = 0$ and $u^N = U_{\lambda,0}$. This proves the claim.

As a consequence, as one checks, when $r_\alpha^{-1} d(x_\alpha, \partial M) \to +\infty$ as $\alpha \to +\infty$, the bubble rewrites

$$B_{x_\alpha, r_\alpha}(u^N) = B_{x_\alpha, \lambda r_\alpha}(U_{1,0}) = \eta\left(\frac{\exp_{x_\alpha}^{-1}(\cdot)}{\tilde{r}_\alpha}\right) \alpha_{n,k} \left(\frac{\lambda r_\alpha}{\lambda^2 r_\alpha^2 + d_g(\cdot, x_\alpha)^2}\right)^{\frac{n-2k}{2}}.$$

We fix $N \in \{1, ..., d\}$. We claim that $(r_\alpha^N)^{-1} d(x_\alpha^N, \partial M) \to +\infty$ as $\alpha \to +\infty$. We argue by contradiction and we assume that the limit is finite. We argue as in the case above. Up to rescaling, and using the boundary chart (3.7), we get that u_α goes to u^N strongly as $\alpha \to +\infty$ in $L^{2_k^\sharp}_{loc}(\mathbb{R}^n \setminus \mathcal{S})$, where $\mathcal{S}$ is finite. Therefore u^N is a nonnegative nonzero weak solution to (3.3), contradicting Lemma 3 in Ge-Wei-Zhou [**7**]. Therefore the limit is infinite and we are back to the previous case.

All these steps prove Theorem 3.2.

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Part 2

Asymptotic Analysis of a Hardy-Sobolev elliptic equation with vanishing singularity

CHAPTER 4

Blow-up Analysis For a Sequence of Solutions of the Critical Hardy-Sobolev Equations

4.1. Introduction

Let Ω be a bounded smooth oriented domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$. We define the Sobolev space $H^2_{1,0}(\Omega)$ as the completion of the space $C^\infty_c(\Omega)$, the space of compactly supported smooth functions in Ω, with respect to the norm

$$\|u\|^2_{H^2_{1,0}(\Omega)} = \int_\Omega |\nabla u|^2 \, dx$$

We let $2^* := \frac{2n}{n-2}$ be the critical Sobolev exponent for the embeding $H^2_{1,0}(\Omega) \hookrightarrow L^p(\Omega)$. Namely, the embedding is defined and continuous for $1 \leq p \leq 2^*$, and it is compact iff $1 \leq p < 2^*$. Let $a \in C^1(\overline{\Omega})$ be such that the operator $\Delta + a$ is coercive in Ω, that is, there exists a constant $A_0 > 0$ such that for all $\varphi \in H^2_{1,0}(\Omega)$

$$\int_\Omega \left(|\nabla\varphi|^2 + a\varphi^2\right) \, dx \geq A_0 \int_\Omega \varphi^2 \, dx \tag{4.1}$$

Solutions $u \in C^2(\overline{\Omega})$ to the problem

$$\begin{cases} \Delta u + a(x)u = u^{2^*-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

(often referred to as "Brezis-Nirenberg problem") are critical points of the functional

$$u \mapsto \frac{\int_\Omega \left(\, |\nabla u|^2 + au^2 \, \right) dx}{\left(\int_\Omega |u|^{2^*} \, dx\right)^{2/2^*}},$$

and a natural way to obtain such critical points is to find minimizers to this functional, that is to prove that

$$\mu_a(\Omega) = \inf_{u \in H^2_{1,0}(\Omega)\setminus\{0\}} \frac{\int_\Omega \left(\, |\nabla u|^2 + au^2 \, \right) dx}{\left(\int_\Omega |u|^{2^*} \, dx\right)^{2/2^*}} \tag{4.2}$$

is achieved. There is a huge and extensive litterature on this problem, starting with the pioneering article of Brezis-Nirenberg [**4**] in which the authors completely solved the question of existence of minimizers for $\mu_a(\Omega)$ when $a \equiv C^{st}$ and $n \geq 4$ for any domain, and $n = 3$ for a ball. Their analysis took inspiration from the contributions of Aubin [**2**] in the resolution of the Yamabe problem. The case when a is arbitrary and $n = 3$ was solved by Druet [**5**] using blowup analysis.

In [**10**], Ghoussoub-Yuan suggested to approach the minimisation problem by adding a singularity in the equation as follows. For any $s \in [0, 2)$, we define

$$2^*(s) := \frac{2(n-s)}{n-2}$$

so that $2^* = 2^*(0)$. Weak solutions $u \in H^2_{1,0}(\Omega)\backslash\{0\}$ to the problem

$$\begin{cases} \Delta u + a(x)u = \frac{u^{2^*(s)-1}}{|x|^s} & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Note here that $0 \in \partial\Omega$ is a boundary point. Such solutions can be achieved as minimizers for the problem

$$\mu_{s,a}(\Omega) = \inf_{u \in H^2_{1,0}(\Omega)\backslash\{0\}} \frac{\int\limits_\Omega \left(|\nabla u|^2 + au^2 \right) dx}{\left(\int\limits_\Omega \frac{|u|^{2^*(s)}}{|x|^s} \, dx \right)^{2/2^*(s)}} \qquad \text{for } s \in (0,2) \tag{4.3}$$

Consider a sequence of positive real numbers $(s_\epsilon)_{\epsilon>0}$ such that $\lim\limits_{\epsilon\to 0} s_\epsilon = 0$. We let $(u_\epsilon)_{\epsilon>0} \in C^2\left(\overline{\Omega}\backslash\{0\}\right) \cap C^1\left(\overline{\Omega}\right)$ such that

$$\begin{cases} \Delta u_\epsilon + a u_\epsilon = \frac{u_\epsilon^{2^*(s_\epsilon)-1}}{|x|^{s_\epsilon}} & \text{in } \Omega, \\ u_\epsilon > 0 & \text{in } \Omega, \\ u_\epsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

Moreover, we assume that the (u_ϵ)'s are of minimal energy type in the sense that

$$\frac{\int\limits_\Omega \left(|\nabla u_\epsilon|^2 + a u_\epsilon^2 \right) dx}{\left(\int\limits_\Omega \frac{|u_\epsilon|^{2^*(s_\epsilon)}}{|x|^s} \, dx \right)^{2/2^*(s_\epsilon)}} = \mu_{s_\epsilon,a}(\Omega) \leq \frac{1}{K(n,0)} + o(1) \tag{4.5}$$

as $\epsilon \to 0$, where $K(n, 0) > 0$ is the best constant in the Sobolev embedding defined in (4.6). Indeed, it follows from Ghoussoub-Robert [**8, 9**] that such a family $(u_\epsilon)_\epsilon$ exists if the the mean curvature of $\partial\Omega$ at 0 is negative.

In this chapter, we are interested here in studying the asymptotic behavior of the sequence $(u_\epsilon)_{\epsilon>0}$ as $\epsilon \to 0$. As proved in Proposition 4.3.2, if the weak limit u_0 of $(u_\epsilon)_\epsilon$ in $H^2_{1,0}(\Omega)$ is nontrivial, then the convergence is indeed strong and u_0 is a minimizer of $\mu_a(\Omega)$. We are dealing here with the more delicate case $u_0 \equiv 0$, in which blow-up necessarily occurs. In the spirit of the C^0-theory of Druet-Hebey-Robert [**6**], our first result is the following:

Theorem 4.1. *Let Ω be a bounded smooth oriented domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$, and let $a \in C^1(\overline{\Omega})$ be such that the operator $\Delta + a$ is coercive in Ω. Let $(s_\epsilon)_{\epsilon>0} \in (0, 2)$ be a sequence such that $\lim\limits_{\epsilon\to 0} s_\epsilon = 0$. Suppose that the sequence $(u_\epsilon)_{\epsilon>0} \in H^2_{1,0}(\Omega)$, where for each $\epsilon > 0$, u_ϵ satisfies* (4.4) *and* (4.5)*, is a* blowup sequence*, i.e*

$$u_\epsilon \rightharpoonup 0 \qquad \textit{weakly in } H^2_{1,0}(\Omega) \qquad \textit{as } \epsilon \to 0$$

Then, there exists $C>0$ such that for all $\epsilon>0$

$$u_\epsilon(x) \le C\left(\frac{\mu_\epsilon}{\mu_\epsilon^2+|x-x_\epsilon|^2}\right)^{\frac{n-2}{2}} \qquad \text{for all } x\in\Omega$$

where

$$\mu_\epsilon^{-\frac{n-2}{2}} = u_\epsilon(x_\epsilon) = \max_{x\in\Omega} u_\epsilon(x).$$

With this optimal pointwise control, we are able to obtain more informations on the localization of the blowup point $x_0 := \lim_{\epsilon\to 0} x_\epsilon$ and the blowup parameter $(\mu_\epsilon)_\epsilon$. We let $G^a : \overline{\Omega}\times\overline{\Omega}\setminus\{(x,x) : x\in\overline{\Omega}\} \longrightarrow \mathbb{R}$ is the Green's function of the coercive operator $\Delta + a$ in Ω with Dirichlet boundary conditions. For any $x\in\Omega$ we write G_x^a as:

$$G_x^a(y) = \frac{1}{(n-2)\omega_{n-1}|x-y|^{n-2}} + g_x^a(y)$$

where ω_{n-1} is the area of the $(n-1)$- sphere. In dimension $n=3$ or when $a\equiv 0$, one has that $g_x^a \in C^2(\overline{\Omega}\setminus\{x\})\cap C^{0,\theta}(\Omega)$ for some $0<\theta<1$, and g^a is called the regular part of the Green's function G^a. In particular, when $n=3$ or $a\equiv 0$, $m_x(\Omega,a) := g_x^a(x)$ is defined for all $x\in\Omega$ and is called the mass of the operator $\Delta + a$.

Theorem 4.2. *Let Ω be a bounded smooth oriented domain of $\mathbb{R}^n$, $n\ge 3$, such that $0\in\partial\Omega$, and let $a\in C^1(\overline{\Omega})$ be such that the operator $\Delta+a$ is coercive in Ω. Let $(s_\epsilon)_{\epsilon>0}\in(0,2)$ be a sequence such that $\lim_{\epsilon\to 0} s_\epsilon = 0$. Suppose that the sequence $(u_\epsilon)_{\epsilon>0}\in H_{1,0}^2(\Omega)$, where for each $\epsilon>0$, u_ϵ satisfies* (4.4) *and* (4.5)*, is a* blowup sequence, *i.e*

$$u_\epsilon \rightharpoonup 0 \qquad \text{weakly in } H_{1,0}^2(\Omega) \qquad \text{as } \epsilon\to 0$$

We let $(\mu_\epsilon)_\epsilon\in(0,+\infty)$ and $(x_\epsilon)_\epsilon\in\Omega$ be such that

$$\mu_\epsilon^{-\frac{n-2}{2}} = u_\epsilon(x_\epsilon) = \max_{x\in\Omega} u_\epsilon(x).$$

We define $x_0 := \lim_{\epsilon\to 0} x_\epsilon$ and we assume that

$$x_0\in\Omega \text{ is an interior point.}$$

Then

$$\lim_{\epsilon\to 0}\frac{s_\epsilon}{\mu_\epsilon^2} = 2^* K(n,0)^{\frac{2^*}{2^*-2}} d_n\ a(x_0) \qquad \text{for } n\ge 5$$
$$\lim_{\epsilon\to 0}\frac{s_\epsilon}{\mu_\epsilon^2\log(1/\mu_\epsilon)} = 256\omega_3 K(4,0)^2\ a(x_0) \qquad \text{for } n=4$$
$$\lim_{\epsilon\to 0}\frac{s_\epsilon}{\mu_\epsilon^{n-2}} = -nb_n^2 K(n,0)^{n/2} g_{x_0}^a(x_0) \qquad \text{for } n=3 \text{ or } a\equiv 0.$$

where $g_{x_0}^a(x_0)$ the mass at the point $x_0\in\Omega$ for the operator $\Delta+a$, where

$$d_n = \int_{\mathbb{R}^n} \frac{1}{\left(1+\frac{|x|^2}{n(n-2)}\right)^{n-2}}\,dx \quad \text{for } n\ge 5\ ;\ b_n = \int_{\mathbb{R}^n}\frac{1}{\left(1+\frac{|x|^2}{n(n-2)}\right)^{\frac{n+2}{2}}}\,dx$$

and ω_3 is the area of the 3- sphere.

When $x_0\in\partial\Omega$ is a boundary point, we get similar estimates:

Theorem 4.3. *Let Ω be a bounded smooth oriented domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$, and let $a \in C^1(\overline{\Omega})$ be such that the operator $\Delta + a$ is coercive in Ω. Let $(s_\epsilon)_{\epsilon>0} \in (0,2)$ be a sequence such that $\lim_{\epsilon\to 0} s_\epsilon = 0$. Suppose that the sequence $(u_\epsilon)_{\epsilon>0} \in H^2_{1,0}(\Omega)$, where for each $\epsilon > 0$, u_ϵ satisfies (4.4) and (4.5), is a* blowup sequence, *i.e*

$$u_\epsilon \rightharpoonup 0 \qquad \text{weakly in } H^2_{1,0}(\Omega) \qquad \text{as } \epsilon \to 0$$

We let $(\mu_\epsilon)_\epsilon \in (0,+\infty)$ and $(x_\epsilon)_\epsilon \in \Omega$ be such that

$$\mu_\epsilon^{-\frac{n-2}{2}} = u_\epsilon(x_\epsilon) = \max_{x\in\Omega} u_\epsilon(x).$$

Assume that

$$\lim_{\epsilon\to 0} x_\epsilon = x_0 \in \partial\Omega.$$

Then

(1) *If $n = 3$ or $a \equiv 0$, then as $\epsilon \to 0$*

$$\lim_{\epsilon\to 0} \frac{s_\epsilon d(x_\epsilon, \partial\Omega)^{n-2}}{\mu_\epsilon^{n-2}} = \frac{n^{n-1}(n-2)^{n-1}K(n,0)^{n/2}\omega_{n-1}}{2^{n-2}}.$$

Moreover, $d(x_\epsilon, \partial\Omega) = (1 + o(1))|x_\epsilon|$ as $\epsilon \to 0$. In particular $x_0 = 0$.

(2) *If $n = 4$. Then as $\epsilon \to 0$*

$$\frac{s_\epsilon}{4}\left(K(4,0)^{-2} + o(1)\right) - \left(\frac{\mu_\epsilon}{d(x_\epsilon, \partial\Omega)}\right)^2 (32\omega_3 + o(1)) = \mu_\epsilon^2 \log\left(\frac{d(x_\epsilon, \partial\Omega)}{\mu_\epsilon}\right)[64\omega_3 a(x_0) + o(1)]$$

(3) *If $n \geq 5$. Then as $\epsilon \to 0$*

$$\frac{s_\epsilon(n-2)}{2n}\left(K(n,0)^{-n/2} + o(1)\right) - \left(\frac{\mu_\epsilon}{d(x_\epsilon, \partial\Omega)}\right)^{n-2}\left(\frac{n^{n-2}(n-2)^n\omega_{n-1}}{2^{n-1}} + o(1)\right) = \mu_\epsilon^2\left[d_n a(x_0) + o(1)\right]$$

where

$$d_n = \int_{\mathbb{R}^n} \frac{1}{\left(1 + \frac{|x|^2}{n(n-2)}\right)^{n-2}}\, dx \quad \text{for } n \geq 5$$

Theorem 4.3 is a particular case of Theorem 4.10 proved in Section 4.8.

The main difficulty in our analysis is due to the natural singularity at $0 \in \partial\Omega$. Indeed, there is a balance between two facts. First, since $s_\epsilon > 0$, this singularity exists and has an influence on the analysis, and in particular on the Pohozaev identity (see the statement of Theorem 4.2). But, second, since $s_\epsilon \to 0$, the singularity should cancel, at least asymptotically. In this perspective, our results are twofolds.

Theorem 4.1 asserts that the pointwise control is the same as the control of the classical problem with $s_\epsilon = 0$: however, to prove this result, we need to perform a very delicate analysis of the blowup with the perturbation $s_\epsilon > 0$, even for the initial steps that are usually standard in the case $s_\epsilon = 0$ (these are Sections 4.4 and 4.5).

The influence and the role of $s_\epsilon > 0$ is much more striking in Theorems 4.2 and 4.3. Compared to the case $s_\epsilon = 0$, the Pohozaev identity (see Section 4.7) enjoys an additional term involving s_ϵ that is present in the statement of Theorems 4.2 and 4.3. Heuristically, this is due to the fact that the limiting equation $\Delta u = |x|^{-s}u^{2^*(s)-1}$ is not invariant under the action of the translations when $s > 0$.

Some classical references for the blow-up analysis of nonlinear critical elliptic pdes are Rey [**16**], Adimurthi- Pacella-Yadava [**1**], Han [**11**], Hebey-Vaugon [**13**] and Khuri-Marques-Schoen [**15**]. The analysis of the 3D problem by Druet [**5**] and the monograph [**6**] by Druet-Hebey-Robert were important sources of inspiration.

This chapter is organized as follows. In Section 4.2, we recall and prove some general facts on Hardy-Sobolev inequalities. In Section 4.3, we prove a few useful general and classical statements. Section 4.4 is a long section devoted to the proof of convergence to a ground state up to rescaling. In Section 4.5, we perform a delicate blow-up analysis to get a first pointwise control on u_ϵ. The optimal control of Theorem 4.1 is proved in Section 4.6. With the pointwise control of Theorem 4.1, we are able to estimate the maximum of the u_ϵ's when the blowup point is in the interior of the domain (Section 4.7) or on the boundary (Section 4.8).

4.2. Some results on Hardy, Sobolev and Hardy-Sobolev inequalities on $\mathbb{R}^n$

The space $\mathscr{D}^{1,2}(\mathbb{R}^n)$ is defined as the completion of the space $C_c^\infty(\mathbb{R}^n)$, the space of compactly supported smooth functions in $\mathbb{R}^n$, with respect to the norm

$$\|u\|^2_{\mathscr{D}^{1,2}} = \int_{\mathbb{R}^n} |\nabla u|^2\, dx$$

The embedding $\mathscr{D}^{1,2}(\mathbb{R}^n) \hookrightarrow L^{2^*}(\mathbb{R}^n)$ is continuous, and we denote the best constant of this embedding by $K(n,0)$ which can be characterised as

$$\frac{1}{K(n,0)} = \inf_{u\in\mathscr{D}^{1,2}(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n} |\nabla u|^2\, dx}{\left(\int_{\mathbb{R}^n} |u|^{2^*}\, dx\right)^{2/2^*}} \tag{4.6}$$

We have for all $u \in \mathscr{D}^{1,2}(\mathbb{R}^n)$

$$\left(\int_{\mathbb{R}^n} |u(x)|^{2^*}\, dx\right)^{2/2^*} \leq K(n,0) \int_{\mathbb{R}^n} |\nabla u|^2\, dx \qquad \textit{The Sobolev inequality} \tag{4.7}$$

We start with the following well known results. Proofs are included for completeness

Lemma 4.2.1.

(i) *For any* $u \in \mathscr{D}^{1,2}(\mathbb{R}^n)$ *one has*

$$\int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^2}\, dx \leq \left(\frac{2}{n-2}\right)^2 \int_{\mathbb{R}^n} |\nabla u|^2\, dx \qquad \textit{The Hardy Inequality} \tag{4.8}$$

(ii) *There exists a constant* $C_{HS} > 0$ *such that for all* $u \in \mathscr{D}^{1,2}(\mathbb{R}^n)$ *one has*

$$\left(\int_{\mathbb{R}^n} \frac{|u(x)|^{2^*(s)}}{|x|^s}\, dx\right)^{2/2^*(s)} \leq C_{HS} \int_{\mathbb{R}^n} |\nabla u|^2\, dx \qquad \textit{The Hardy-Sobolev Inequality} \tag{4.9}$$

Proof. By density it is enough to consider $u \in C_c^\infty(\mathbb{R}^n)$. For a $x \in \mathbb{R}^n$ we have

$$
\begin{aligned}
|u(x)|^2 &= -\int_1^{+\infty} \left(|u(tx)|^2\right)' dt \\
&= -2 \int_1^{+\infty} u(tx) \left\langle \nabla u(tx), x \right\rangle \, dt
\end{aligned}
$$

By Fubini theorem we then have

$$
\begin{aligned}
\int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^2} \, dx &= -2 \int_1^{+\infty} \int_{\mathbb{R}^n} \frac{u(tx)}{|x|^2} \left\langle \nabla u(tx), x \right\rangle \, dx \, dt \\
&= -2 \int_1^{+\infty} \int_{\mathbb{R}^n} u(tx) \left\langle \nabla u(tx), \frac{x}{|x|^2} \right\rangle \, dx \, dt \\
&= -2 \int_1^{+\infty} \frac{1}{t^{n-1}} \, dt \times \int_{\mathbb{R}^n} u(x) \left\langle \nabla u(x), \frac{x}{|x|^2} \right\rangle \, dx \\
&= -\frac{2}{n-2} \int_{\mathbb{R}^n} \frac{u(x)}{|x|} \left\langle \nabla u(x), \frac{x}{|x|} \right\rangle \, dx
\end{aligned}
$$

Using Hölder inequality we obtain that

$$
\int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^2} \, dx \leq \left(\frac{2}{n-2}\right) \left(\int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^2} \, dx \right)^{1/2} \left(\int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx \right)^{1/2}
$$

Therefore

$$
\int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^2} \, dx \leq \left(\frac{2}{n-2}\right)^2 \int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx
$$

and we have the *Hardy inequality*.

Now for $u \in \mathscr{D}^{1,2}(\mathbb{R}^n)$ we have

$$
\begin{aligned}
\int_{\mathbb{R}^n} \frac{|u(x)|^{2^*(s)}}{|x|^s} \, dx &= \int_{\mathbb{R}^n} \frac{|u(x)|^s}{|x|^s} |u(x)|^{2^*(s)-s} \, dx \\
&\leq \left(\int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^2} \, dx \right)^{s/2} \left(\int_{\mathbb{R}^n} |u(x)|^{2^*} dx \right)^{\frac{2-s}{2}} \qquad \text{by Hölder inequality} \\
&\leq K(n,0)^{\frac{2-s}{2} \frac{n}{n-2}} \left(\frac{2}{n-2}\right)^s \left(\int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx \right)^{2^*(s)/2}
\end{aligned}
$$

by *Hardy inequality* (4.8) and *Sobolev inequality* (4.7)

Hence we have obtained that for all $u \in \mathscr{D}^{1,2}(\mathbb{R}^n)$

$$\left(\int_{\mathbb{R}^n} \frac{|u(x)|^{2^*(s)}}{|x|^s} \, dx \right)^{2/2^*(s)} \leq \left[K(n,0)^{\frac{n}{2}\left(\frac{2-s}{n-s}\right)} \left(\frac{2}{n-2} \right)^{2s/2^*(s)} \right] \int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx$$

This completes the lemma. □

We let

$$\frac{1}{K(n,s)} = \inf_{u \in \mathscr{D}^{1,2}(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} |\nabla u|^2 \, dx}{\left(\int_{\mathbb{R}^n} \frac{|u|^{2^*(s)}}{|x|^s} \, dx \right)^{2/2^*(s)}} \tag{4.10}$$

Proposition 4.2.1.

$$\lim_{s \to 0} K(n,s) = K(n,0)$$

PROOF. Let $u \in \mathscr{D}^{1,2}(\mathbb{R}^n)$. In lemma 4.2.1 we have obtained

$$\left(\int_{\mathbb{R}^n} \frac{|u(x)|^{2^*(s)}}{|x|^s} \, dx \right)^{2/2^*(s)} \leq \left[K(n,0)^{\frac{n}{2}\left(\frac{2-s}{n-s}\right)} \left(\frac{2}{n-2} \right)^{2s/2^*(s)} \right] \int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx$$

So

$$K(n,s) \leq K(n,0)^{\frac{n}{2}\left(\frac{2-s}{n-s}\right)} \left(\frac{2}{n-2} \right)^{2s/2^*(s)}$$

Letting $s \to 0$, one has

$$\limsup_{s \to 0} K(n,s) \leq K(n,0)$$

Next by Fatou's lemma

$$\int_{\mathbb{R}^n} |u(x)|^{2^*} \, dx \leq \liminf_{s \to 0} \int_{\mathbb{R}^n} \frac{|u(x)|^{2^*(s)}}{|x|^s} \, dx$$

And so

$$\int_{\mathbb{R}^n} |u(x)|^{2^*} \, dx \leq \liminf_{s \to 0} \int_{\mathbb{R}^n} \frac{|u(x)|^{2^*(s)}}{|x|^s} \, dx \leq \liminf_{s \to 0} \left[K(n,s)^{2^*(s)/2} \left(\int_{\mathbb{R}^n} |\nabla u|^2 \, dx \right)^{2^*(s)/2} \right],$$

$$\left(\int_{\mathbb{R}^n} |u(x)|^{2^*} \, dx \right)^{2/2^*} \leq \left(\liminf_{s \to 0} K(n,s) \right) \int_{\mathbb{R}^n} |\nabla u|^2 \, dx$$

Therefore

$$K(n,0) \leq \liminf_{s \to 0} K(n,s)$$

Hence

$$\lim_{s \to 0} K(n,s) = K(n,0)$$

□

4.3. Hardy-Sobolev inequality on Ω and the case of a nonzero weak limit

Recall that Ω is a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$. We then have the following useful proposition:

Proposition 4.3.1.

$$\lim_{s\to 0} \mu_{s,a}(\Omega) = \mu_a(\Omega)$$

Proof. Let $u \in H^2_{1,0}(\Omega)\backslash\{0\}$. One has

$$\begin{aligned}\int_\Omega \frac{|u(x)|^{2^*(s)}}{|x|^s}\,dx &= \int_\Omega \frac{|u(x)|^s}{|x|^s}|u(x)|^{2^*(s)-s}\,dx \\ &\leq \left(\int_\Omega \frac{|u(x)|^2}{|x|^2}\,dx\right)^{s/2}\left(\int_{\mathbb{R}^n}|u(x)|^{2^*}dx\right)^{\frac{2-s}{2}} \qquad \text{by Hölder inequality} \\ &\leq \left(\frac{2}{n-2}\right)^s\left(\int_\Omega |\nabla u(x)|^2\,dx\right)^{s/2}\left(\int_\Omega |u(x)|^{2^*}dx\right)^{\frac{2-s}{2}} \qquad \text{by } \textit{Hardy inequality } (4.8)\end{aligned}$$

So

$$\left(\int_\Omega \frac{|u(x)|^{2^*(s)}}{|x|^s}\,dx\right)^{2/2^*(s)} \leq \left(\frac{2}{n-2}\right)^{2s/2^*(s)}\left(\int_\Omega |\nabla u(x)|^2\,dx\right)^{s/2^*(s)}\left(\int_\Omega |u(x)|^{2^*}dx\right)^{\frac{2-s}{2^*(s)}}$$

And hence

$$\frac{\int_\Omega \left(|\nabla u|^2 + au^2\right)\,dx}{\left(\int_\Omega |u|^{2^*}\,dx\right)^{2/2^*}} \leq \frac{\int_\Omega \left(|\nabla u|^2 + au^2\right)\,dx}{\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}\,dx\right)^{2/2^*(s)}}\left[\left(\frac{2}{n-2}\right)^{2s/2^*(s)}\left(\int_\Omega |\nabla u(x)|^2\,dx\right)^{s/2^*(s)}\left(\int_\Omega |u(x)|^{2^*}dx\right)\right.$$

which, by *Sobolev inequality* (4.7) gives that for all $u \in H^2_{1,0}(\Omega)\backslash\{0\}$

$$\frac{\int_\Omega \left(|\nabla u|^2 + au^2\right)\,dx}{\left(\int_\Omega |u|^{2^*}\,dx\right)^{2/2^*}} \leq \frac{\int_\Omega \left(|\nabla u|^2 + au^2\right)\,dx}{\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}\,dx\right)^{2/2^*(s)}}\left(\frac{1}{K(n,0)^{1/2^*}}\frac{2}{n-2}\right)^{s\frac{n-2}{n-s}}$$

So

$$\mu_a(\Omega) \leq \mu_{s,a}(\Omega)\left(\frac{1}{K(n,0)^{1/2^*}}\frac{2}{n-2}\right)^{s\frac{n-2}{n-s}}$$

Passing to limits as $s \to 0$, one obtains that

$$\mu_a(\Omega) \leq \liminf_{s\to 0} \mu_{s,a}(\Omega)$$

Let $u \in H^2_{1,0}(\Omega)\backslash\{0\}$. By Fatou's lemma one has

$$\int_\Omega |u(x)|^{2^*}\ dx \le \liminf_{s\to 0} \int_\Omega \frac{|u(x)|^{2^*(s)}}{|x|^s}\ dx \le \liminf_{s\to 0} \left(\frac{1}{\mu_{s,a}(\Omega)} \int_\Omega \left(|\nabla u|^2 + au^2\right)\ dx \right)^{2^*(s)/2},$$

$$\left(\int_\Omega |u(x)|^{2^*}\ dx \right)^{2/2^*} \le \liminf_{s\to 0} \frac{1}{\mu_{s,a}(\Omega)} \int_\Omega \left(|\nabla u|^2 + au^2\right)\ dx$$

Therefore

$$\liminf_{s\to 0} \frac{1}{\mu_{s,a}(\Omega)} \ge \frac{1}{\mu_a(\Omega)}$$

And so

$$\limsup_{s\to 0} \mu_{s,a}(\Omega) \le \mu_a(\Omega)$$

Hence

$$\lim_{s\to 0} \mu_{s,a}(\Omega) = \mu_a(\Omega)$$

□

We now prove the following proposition for nonzero weak limits:

Proposition 4.3.2. *Let Ω be a bounded smooth oriented domain of $\mathbb{R}^n$, $n \ge 3$, such that $0 \in \partial\Omega$. Let $a \in C^1(\overline{\Omega})$ be such that the operator $\Delta + a$ is coercive in Ω Let $(u_\epsilon)_{\epsilon>0} \in C^2\left(\overline{\Omega}\backslash\{0\}\right) \cap C^1\left(\overline{\Omega}\right)$ be as in (4.4) and (4.5). Then there exists $u_0 \in H^2_{1,0}(\Omega)$ such that $u_\epsilon \rightharpoonup u_0$ weakly in $H^2_{1,0}(\Omega)$ as $\epsilon \to 0$. Indeed, $u_0 \in C^2\left(\overline{\Omega}\backslash\{0\}\right) \cap C^1\left(\overline{\Omega}\right)$ is a solution to*

$$\begin{cases} \Delta u_0 + au_0 = u_0^{2^*-1} & \text{in } \Omega \\ u_0 \ge 0 & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega \end{cases}$$

If $u_0 \neq 0$, then $u_0 > 0$ in Ω and

$$\mu_a(\Omega) = \frac{\int_\Omega \left(\ |\nabla u_0|^2 + au_0^2\ \right) dx}{\left(\int_\Omega |u_0|^{2^*}\ dx\right)^{2/2^*}}$$

Therefore $\mu_a(\Omega)$ is attained. Further $u_\epsilon \to u_0$ in $H^2_{1,0}(\Omega)$, as $\epsilon \to 0$.

Proof. First, from the coercivity of the operator $\Delta + a$ it follows that the sequence $(u_\epsilon)_{\epsilon>0}$ is bounded in $H^2_{1,0}(\Omega)$, i.e

$$\|u_\epsilon\|_{H^2_{1,0}(\Omega)} = O(1) \qquad \text{as } \epsilon \to 0 \tag{4.11}$$

Then from the weak compactness of the unit ball in $H^2_{1,0}(\Omega)$ it follows that there exists $u_0 \in H^2_{1,0}(\Omega)$ such that upto a subsequence, as $\epsilon \to 0$

$$u_\epsilon \rightharpoonup u_0 \qquad \text{weakly in } H^2_{1,0}(\Omega)$$

And so as $\epsilon \to 0$

$$\left\{\begin{array}{lll} u_\epsilon \rightharpoonup u_0 & \text{weakly in } L^{2^*}(\Omega) & \\ u_\epsilon \to u_0 & \text{strongly in } L^p(\Omega) & \text{for } 1 < p < 2^* \\ u_\epsilon(x) \to u_0(x) & a.e\ x \text{ in } \Omega & \end{array}\right. \tag{4.12}$$

In particular, for any $\varphi \in C_c^\infty(\Omega)$

$$\int_\Omega (\langle \nabla u_\epsilon, \nabla\varphi\rangle + a u_\epsilon \varphi)\, dx \longrightarrow \int_\Omega (\langle \nabla u_0, \nabla\varphi\rangle + a u_0 \varphi)\, dx \qquad \text{as } \epsilon \to 0$$

One has that $\dfrac{u_\epsilon^{2^*(s_\epsilon)-1}}{|x|^{s_\epsilon}} \to u_0^{2^*-1}$ $a.e$ in Ω as $\epsilon \to 0$ and the sequence $\left(\dfrac{u_\epsilon^{2^*(s_\epsilon)-1}}{|x|^{s_\epsilon}}\right)_{\epsilon>0}$ is bounded in $L^{\delta_0 \frac{2^*}{2^*-1}}(\Omega)$ for some $\delta_0 < 1$ sufficiently small. So by integration theory

$$\int_\Omega \frac{u_\epsilon^{2^*(s_\epsilon)-1}}{|x|^{s_\epsilon}}\varphi\ dx \longrightarrow \int_\Omega u_0^{2^*-1}\varphi\ dx$$

Therefore u_0 is a weak solution of the equation

$$\left\{\begin{array}{ll} \Delta u_0 + a u_0 = u_0^{2^*-1} & \text{in } \Omega \\ u_0 \geq 0 & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega \end{array}\right. \tag{4.13}$$

It follows by the regularity result in *Ghoussoub-Robert* [**8**], [**9**] that $u_0 \in C^2\left(\overline{\Omega}\backslash\{0\}\right) \cap C^1\left(\overline{\Omega}\right)$. Multiplying both sides of eqn (4.13) by u_0 gives that

$$\int_{\mathbb{R}^n} \left(|\nabla u_0|^2 + a u_0^2\right) dx = \int_\Omega u_0^{2^*}\, dx$$

So if $u_0 \neq 0$ it follows from the definition (4.2) of $\mu_a(\Omega)$ that

$$\int_\Omega u_0^{2^*}\, dx \geq \mu_a(\Omega)^{\frac{2^*}{2^*-2}}$$

Since $u_\epsilon \rightharpoonup u_0$ weakly in $H_{1,0}^2(\Omega)$ as $\epsilon \to 0$, using the Fatou's lemma one has

$$\int_\Omega |u_0(x)|^{2^*}\ dx \leq \liminf_{\epsilon\to 0} \int_\Omega \frac{|u_0(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\ dx$$

From (4.4) and (4.5), we have

$$\int_\Omega \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\ dx = \mu_{s_\epsilon,a}(\Omega)^{\frac{2^*(s_\epsilon)}{2^*(s_\epsilon)-2}}$$

This together with proposition 4.3.1 gives that

$$\int_\Omega |u_0|^{2^*}\ dx \leq \mu_a(\Omega)^{\frac{2^*}{2^*-2}}$$

Hence

$$\int_\Omega |u_0|^{2^*}\ dx = \mu_a(\Omega)^{\frac{2^*}{2^*-2}}$$

And so

$$\int_{\mathbb{R}^n} \left(|\nabla u_0|^2 + a u_0^2\right) dx = \mu_a(\Omega)^{\frac{2^*}{2^*-2}}$$

Therefore we obtain if $u_0 \neq 0$

$$\frac{\int_\Omega \left(|\nabla u_0|^2 + a u_0^2 \right) dx}{\left(\int_\Omega |u_0|^{2^*} \, dx \right)^{2/2^*}} = \mu_a(\Omega)$$

Let

$$v_\epsilon = u_\epsilon - u_0$$

Then as $\epsilon \to 0$

$$\begin{cases} v_\epsilon \rightharpoonup 0 & \text{weakly in } H_{1,0}^2(\Omega) \\ v_\epsilon \rightharpoonup 0 & \text{weakly in } L^{2^*}(\Omega) \\ v_\epsilon \to 0 & \text{strongly in } L^p(\Omega) \qquad \text{for } \ 1 < p < 2^* \\ v_\epsilon(x) \to 0 & a.e \ x \text{ in } \Omega \end{cases}$$

We have

$$\int_{\mathbb{R}^n} \left(|\nabla u_\epsilon|^2 + a u_\epsilon^2\right) dx = \int_{\mathbb{R}^n} \left(|\nabla u_0|^2 + a u_0^2\right) dx + \int_{\mathbb{R}^n} \left(|\nabla v_\epsilon|^2 + a v_\epsilon^2\right) dx + o(1) \qquad \text{as } \epsilon \to 0$$

If $u_0 \neq 0$ then

$$\mu_{s_\epsilon,a}(\Omega)^{\frac{2^*(s_\epsilon)}{2^*(s_\epsilon)-2}} = \mu_a(\Omega)^{\frac{2^*}{2^*-2}} + \int_{\mathbb{R}^n} |\nabla v_\epsilon|^2 \, dx + o(1) \qquad \text{as } \epsilon \to 0$$

Letting $\epsilon \to 0$ and using proposition 4.3.1 we obtain that

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^n} |\nabla v_\epsilon|^2 \, dx = 0$$

And therefore

$$u_\epsilon \to u_0 \qquad \text{in } H_{1,0}^2(\Omega) \text{ as } \epsilon \to 0$$

$\square$

4.4. Preliminary Blow-up Analysis

We let (u_ϵ) be as in Theorem 4.1. We will say that *blowup occurs* whenever

$$u_\epsilon \rightharpoonup 0 \qquad \text{weakly in } H_{1,0}^2(\Omega) \qquad \text{as } \ \epsilon \to 0$$

We describe the behaviour of such a sequence of solutions (u_ϵ). By regularity, for all ϵ, $u_\epsilon \in C^0(\overline{\Omega})$. We let $x_\epsilon \in \Omega$ and $\mu_\epsilon > 0$ be such that :

$$u_\epsilon(x_\epsilon) = \max_{\overline{\Omega}} u_\epsilon(x) \qquad \text{and} \qquad \mu_\epsilon^{-\frac{n-2}{2}} = u_\epsilon(x_\epsilon) \tag{4.14}$$

This section is devoted to the proof of the following theorem:

Theorem 4.4. *Let Ω be a bounded smooth oriented domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$, and let $a \in C^1(\overline{\Omega})$ be such that the operator $\Delta + a$ is coercive in Ω. Let $(s_\epsilon)_{\epsilon>0} \in (0,2)$ be a sequence such that $\lim_{\epsilon\to 0} s_\epsilon = 0$. Suppose that the sequence $(u_\epsilon)_{\epsilon>0} \in H^2_{1,0}(\Omega)$, where for each $\epsilon > 0$, u_ϵ satisfies (4.4) and (4.5), is a* blowup sequence, *i.e*

$$u_\epsilon \rightharpoonup 0 \qquad \text{weakly in } H^2_{1,0}(\Omega) \qquad \text{as } \epsilon \to 0$$

We let $(x_\epsilon)_\epsilon, (\mu_\epsilon)_\epsilon$ be as in (4.14). Let k_ϵ be such that

$$k_\epsilon := |x_\epsilon|^{s_\epsilon/2} \mu_\epsilon^{\frac{2-s_\epsilon}{2}} \qquad \text{for } \epsilon > 0 \tag{4.15}$$

Then

$$\lim_{\epsilon\to 0} \mu_\epsilon = \lim_{\epsilon\to 0} k_\epsilon = 0 \text{ and } \lim_{\epsilon\to 0} \frac{d(x_\epsilon, \partial\Omega)}{\mu_\epsilon} = \lim_{\epsilon\to 0} \frac{d(x_\epsilon, \partial\Omega)}{k_\epsilon} = +\infty.$$

We rescale and define

$$v_\epsilon(x) := \frac{u_\epsilon(x_\epsilon + k_\epsilon x)}{u_\epsilon(x_\epsilon)} \qquad \text{for } x \in \frac{\Omega - x_\epsilon}{k_\epsilon}$$

Then there exists $v \in C^\infty(\mathbb{R}^n)$ such that $v \not\equiv 0$ and for any $\eta \in C_c^\infty(\mathbb{R}^n)$

$$\eta v_\epsilon \rightharpoonup \eta v \qquad \text{weakly in } \mathscr{D}^{1,2}(\mathbb{R}^n) \qquad \text{as } \epsilon \to 0$$

Further for all $x \in \mathbb{R}^n$ $v(x) \leq v(0) = 1$ and it satisfies the equation

$$\begin{cases} \Delta v = v^{2^*-1} & \text{in } \mathbb{R}^n \\ v \geq 0 & \text{in } \mathbb{R}^n \end{cases}$$

One has

$$v(x) = \left(\frac{1}{1 + \frac{|x|^2}{n(n-2)}}\right)^{\frac{n-2}{2}} \qquad \text{for } x \in \mathbb{R}^n \qquad \text{and} \quad \int_{\mathbb{R}^n} |\nabla v|^2 \, dx = \left(\frac{1}{K(n,0)}\right)^{\frac{2^*}{2^*-2}} \tag{4.16}$$

Also

$$v_\epsilon \longrightarrow v \qquad \text{in } C^1_{loc}(\mathbb{R}^n) \qquad \text{as } \epsilon \to 0$$

and, moreover upto a subsequence, as $\epsilon \to 0$

$$\left(\frac{\mu_\epsilon}{|x_\epsilon|}\right)^{s_\epsilon} \to 1 \qquad \text{and} \quad \frac{k_\epsilon}{\mu_\epsilon} \to 1 \tag{4.17}$$

The rest of the section is devoted to the proof of Theorem 4.4. It goes through four steps.

Step 1: We claim that

$$\mu_\epsilon = o(1) \text{ and } k_\epsilon = o(1) \qquad \text{as } \epsilon \to 0$$

Proof. We proceed by contradiction. Suppose

$$\lim_{\epsilon\to 0} \mu_\epsilon \neq 0$$

Then by our definition (4.14) this implies that for for all ϵ

$$\|u_\epsilon\|_{L^\infty(\Omega)} \leq C$$

for some positive constant C. Therefore $\dfrac{u_\epsilon^{2^*(s_\epsilon)-1}}{|x|^{s_\epsilon}}$ is uniformly bounded in $L^p(\Omega)$ for some $p > n$. Then from eqn (4.4) and standard elliptic estimates (see for instance [**14**]) it follows that for all ϵ

$$\|u_\epsilon\|_{C^{1,\alpha}(\Omega)} \leq C'$$

for some positive constant C' and $\alpha \in (0,1)$. Hence the sequence (u_ϵ) is precompact in the space $C^1(\overline{\Omega})$. Since $u_\epsilon \rightharpoonup 0$ weakly in $H^2_{1,0}(\Omega)$, therefore $u_\epsilon \to 0$ in $C^1(\overline{\Omega})$, as $\epsilon \to 0$.

From (4.4)and (4.5) we obtain that

$$\int_\Omega \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\, dx = \mu_{s_\epsilon,a}(\Omega)^{\frac{2^*(s_\epsilon)}{2^*(s_\epsilon)-2}}$$

But if $u_\epsilon \to 0$ in $C^1(\overline{\Omega})$ as $\epsilon \to 0$, then this implies that

$$\lim_{\epsilon\to 0} \mu_{s_\epsilon,a}(\Omega) = 0$$

And therefore $\mu_a(\Omega) = 0$, a contradiction since the operator $\Delta + a$ is coercive in Ω. So, we must have that $\lim\limits_{\epsilon\to 0} \mu_\epsilon = 0$. The result for k_ϵ follows from the definition. This ends Step 1. □

We let

$$\mathbb{R}^n_- = \{x \in \mathbb{R}^n : x_1 < 0\}$$

where x_1 is the first coordinate of a generic point in $\mathbb{R}^n$. This space will be the limit space in certain cases after blowup. We describe a parametrisation around a point of the boundary $\partial\Omega$. Let $p \in \partial\Omega$. Then there exists U,V open in $\mathbb{R}^n$ and a smooth diffeomorphism $\mathcal{T} : U \longrightarrow V$ such that upto a rotation of coordinates if necessary

(4.18)

- $0 \in U$ and $p \in V$
- $\mathcal{T}(0) = p$
- $\mathcal{T}(U \cap \{x_1 < 0\}) = V \cap \Omega$
- $\mathcal{T}(U \cap \{x_1 = 0\}) = V \cap \partial\Omega$
- $D_0\mathcal{T} = \mathbb{I}_{\mathbb{R}^n}$. Here $D_x\mathcal{T}$ denotes the differential of $\mathcal{T}$ at the point x and $\mathbb{I}_{\mathbb{R}^n}$ is the identity map on $\mathbb{R}^n$.
- $\mathcal{T}_*(0)\ (e_1) = \nu_p$ where ν_p denotes the outer unit normal vector to $\partial\Omega$ at the point p.
- $\{\mathcal{T}_*(0)(e_2), \cdots, (\mathcal{T}_m)_*(0)(e_n)\}$ forms an orthonormal basis of $T_p\partial\Omega$.

Step 2: We claim that

$$\lim_{\epsilon\to 0} \frac{|x_\epsilon|}{\mu_\epsilon} = +\infty \tag{4.19}$$

PROOF. Suppose on the contrary

$$\frac{|x_\epsilon|}{\mu_\epsilon} = O(1) \qquad \text{as } \epsilon \to 0$$

Then

$$\lim_{\epsilon\to+\infty}|x_\epsilon| = 0$$

Let $\mathcal{T}_0 : U \to V$ be a parametrisation of the boundary as in (4.18) at the point $p = 0$. For all $\epsilon > 0$, we let

$$\tilde{v}_\epsilon(x) = \frac{u_\epsilon \circ \mathcal{T}_0(\mu_\epsilon x)}{u_\epsilon(x_\epsilon)} \qquad \text{for } x \in \frac{U}{\mu_\epsilon} \cap \{x_1 \leq 0\}$$

Step 2.1: For any $\eta \in C_c^\infty(\mathbb{R}^n)$, one has that $\eta\tilde{v}_\epsilon \in \mathscr{D}^{1,2}(\mathbb{R}^n_-)$ for $\epsilon > 0$ sufficiently small. We claim that there exists $\tilde{v}_\eta \in \mathscr{D}^{1,2}(\mathbb{R}^n_-)$ such that upto a subsequence

$$\begin{cases} \eta\tilde{v}_\epsilon \rightharpoonup \tilde{v}_\eta & \text{weakly in } \mathscr{D}^{1,2}(\mathbb{R}^n_-) \text{ as } \epsilon \to 0 \\ \eta\tilde{v}_\epsilon(x) \to \tilde{v}_\eta(x) & a.e\ x \text{ in } \mathbb{R}^n_- \text{ as } \epsilon \to 0 \end{cases}$$

We prove the claim. Let $x \in \mathbb{R}^n_-$, then

$$\nabla\left(\eta\tilde{v}_\epsilon\right)(x) = \tilde{v}_\epsilon(x)\nabla\eta(x) + \frac{\mu_\epsilon}{u_\epsilon(x_\epsilon)}\,\eta(x) D_{(\mu_\epsilon x)}\mathcal{T}_0\left[\nabla u_\epsilon\left(\mathcal{T}_0(\mu_\epsilon x)\right)\right]$$

For any $\theta > 0$, there exists $C(\theta) > 0$ such that for any $a, b > 0$

$$(a+b)^2 \leq C(\theta)a^2 + (1+\theta)b^2$$

With this inequality we then obtain

$$\int_{\mathbb{R}^n_-} \left|\nabla\left(\eta\tilde{v}_\epsilon\right)\right|^2\, dx \leq C(\theta)\int_{\mathbb{R}^n_-}|\nabla\eta|^2\tilde{v}_\epsilon^2\, dx + (1+\theta)\frac{\mu_\epsilon^2}{u_\epsilon^2(x_\epsilon)}\int_{\mathbb{R}^n_-}\eta^2\left|D_{(\mu_\epsilon x)}\mathcal{T}_0\left[\nabla u_\epsilon\left(\mathcal{T}_0(\mu_\epsilon x)\right)\right]\right|^2\, dx$$

Since $D_0\mathcal{T}_0 = \mathbb{I}_{\mathbb{R}^n}$ we have as $\epsilon \to 0$

$$\int_{\mathbb{R}^n_-} \left|\nabla\left(\eta\tilde{v}_\epsilon\right)\right|^2\, dx \leq C(\theta)\int_{\mathbb{R}^n_-}|\nabla\eta|^2\tilde{v}_\epsilon^2\, dx + (1+\theta)\left(1+O(\mu_\epsilon)\right)\frac{\mu_\epsilon^2}{u_\epsilon^2(x_\epsilon)}\int_{\mathbb{R}^n_-}\eta^2\left|\nabla u_\epsilon\left(\mathcal{T}_0(\mu_\epsilon x)\right)\right|^2(1+o(1))dx$$

With Hölder inequality and a change of variables this becomes

(4.20)

$$\int_{\mathbb{R}^n_-} \left|\nabla\left(\eta\tilde{v}_\epsilon\right)\right|^2\, dx \leq C(\theta)\left\|\nabla\eta\right\|^2_{L^n}\left(\int_\Omega u_\epsilon^{2^*}\, dx\right)^{\frac{n-2}{n}} + (1+\theta)\left(1+O(\mu_\epsilon)\right)\int_{\mathbb{R}^n}|\nabla u_\epsilon|^2\, dx$$

Now since $\|u_\epsilon\|_{H^2_{1,0}(\Omega)} = O(1)$ and $\mu_\epsilon \to 0$ as $\epsilon \to 0$, so for $\epsilon > 0$ small enough

$$\|\eta\tilde{v}_\epsilon\|_{\mathscr{D}^{1,2}(\mathbb{R}^n_-)} \leq C_\eta$$

Where C_η is a constant depending on the function η. The claim then follows from the reflexivity of $\mathscr{D}^{1,2}(\mathbb{R}^n_-)$.

Step 2.2: Let $\eta_1 \in C_c^\infty(\mathbb{R}^n)$, $0 \leq \eta_1 \leq 1$ be a smooth cut-off function, such that

$$\eta_1 = \begin{cases} 1 & \text{for } x \in B_0(1) \\ 0 & \text{for } x \in \mathbb{R}^n \backslash B_0(2) \end{cases} \tag{4.21}$$

For any $R > 0$ we let $\eta_R = \eta_1(x/R)$. Then with a diagonal argument we can assume that, upto a subsequence for any $R > 0$, there exists $\tilde{v}_R \in \mathscr{D}^{1,2}(\mathbb{R}^n_-)$ such that

$$\begin{cases} \eta_R\tilde{v}_\epsilon \rightharpoonup \tilde{v}_R & \text{weakly in } \mathscr{D}^{1,2}(\mathbb{R}^n_-) \text{ as } \epsilon \to 0 \\ \eta_R\tilde{v}_\epsilon(x) \to \tilde{v}_R(x) & a.e\ x \text{ in } \mathbb{R}^n_- \text{ as } \epsilon \to 0 \end{cases}$$

Since $\|\nabla\eta_R\|_n^2 = \|\nabla\eta_1\|_n^2$ for all $R>0$, letting $\epsilon\to 0$ in (4.20) we obtain that

$$\int_{\mathbb{R}^n_-} |\nabla\tilde{v}_R|^2\,dx \le C \qquad \text{for all } R>0$$

where C is a constant independent of R. So there exists $\tilde{v}\in\mathscr{D}^{1,2}(\mathbb{R}^n_-)$ such that

$$\begin{cases} \tilde{v}_R \rightharpoonup \tilde{v} & \text{weakly in } \mathscr{D}^{1,2}(\mathbb{R}^n_-) \text{ as } R\to\infty \\ \tilde{v}_R(x)\to\tilde{v}(x) & a.e\ x \text{ in } \mathbb{R}^n_- \text{ as } R\to\infty \end{cases}$$

Step 2.3: We claim that $\tilde{v}\in C^1(\overline{\mathbb{R}^n_-})$ and it satisfies weakly the equation

$$\begin{cases} \Delta\tilde{v} = \tilde{v}^{2^*-1} & \text{in } \mathbb{R}^n_- \\ \tilde{v}=0 & \text{on } \{x_1=0\} \end{cases}$$

We prove the claim. For $i,j=1,\dots,n$, we let $g_{ij}=(\partial_i\mathcal{T}_0,\partial_j\mathcal{T}_0)$, the metric induced by the chart $\mathcal{T}_0$ on the domain $U\cap\{x_1<0\}$ and let Δ_g denote the Laplace-Beltrami operator with respect to the metric g. We let

$$\tilde{g}_\epsilon = g\left(\mu_\epsilon x\right)$$

From eqn (4.4) it follows that for any $\epsilon>0$ and $R>0$, $\eta_R\tilde{v}_\epsilon$ satisfies weakly the equation

$$\begin{cases} \Delta\left(\eta_R\tilde{v}_\epsilon\right)+\mu_\epsilon^2\left(a\circ\mathcal{T}_0(\mu_\epsilon x)\right)\left(\eta_R\tilde{v}_\epsilon\right) = \frac{(\eta_R\tilde{v}_\epsilon)^{2^*(s_\epsilon)-1}}{\left|\frac{\mathcal{T}_0(\mu_\epsilon x)}{\mu_\epsilon}\right|^{s_\epsilon}} & \text{in } B_0(R)\cap\{x_1<0\} \\ \eta_R\tilde{v}_\epsilon=0 & \text{on } B_0(R)\cap\{x_1=0\} \end{cases} \tag{4.22}$$

Now $0\le\tilde{v}_\epsilon\le 1$ and from the properties of the boundary chart $\mathcal{T}_0$, it follows that for any $p>1$ there exists a constant C_p such that

$$\int_{B_0(R)\cap\{x_1<0\}} \left[\frac{(\eta_R\tilde{v}_\epsilon)^{2^*(s_\epsilon)-1}}{\left|\frac{\mathcal{T}_0(\mu_\epsilon x)}{\mu_\epsilon}\right|^{s_\epsilon}}\right]^p dx \le C_p \int_{B_0(R)\cap\{x_1<0\}} \frac{1}{|x|^{s_\epsilon p}}\,dx$$

So the right hand side of equation (4.22) is uniformly bounded in L^p for some $p>n$. Then from standard elliptic estimates (see for instance [**14**]) it follows that the sequence $(\eta_R\tilde{v}_\epsilon)_{\epsilon>0}$ is bounded in $C^{1,\alpha_0}\left(B_0(R)\ \cap\{x_1\le 0\}\right)$ for some $\alpha_0\in(0,1)$. So by *Arzela-Ascoli's theorem* one has that $\tilde{v}_R\in C^{1,\alpha}\left(B_0(R/2)\ \cap\{x_1\le 0\}\right)$ for $0<\alpha<\alpha_0$, and that, up to a subsequence

$$\lim_{\epsilon\to 0}\eta_R\tilde{v}_\epsilon = \tilde{v}_R \qquad \text{in } C^{1,\alpha}\left(B_0(R/4)\ \cap\{x_1\le 0\}\right)$$

for $0<\alpha<\alpha_0$. And therefore

$$\tilde{v}_R\equiv 0 \qquad \text{on } B_0(R/4)\ \cap\{x_1=0\} \tag{4.23}$$

Letting $\epsilon\to 0$ in eqn (4.22) gives that $\tilde{v}_R$ satisfies weakly the equation

$$\begin{cases} \Delta\tilde{v}_R = \tilde{v}_R^{2^*-1} & \text{in } B_0(R/4)\cap\{x_1\le 0\} \\ \tilde{v}_R=0 & \text{on } B_0(R/4)\cap\{x_1=0\} \end{cases} \tag{4.24}$$

Again we have that: $0\le\tilde{v}_R\le 1$, then again from standard elliptic estimates and applying the *Arzela-Ascoli's theorem* it follows that $\tilde{v}\in C^1(\overline{\mathbb{R}^n_-})$ and $\lim\limits_{R\to+\infty}\tilde{v}_R=\tilde{v}$

in $C^1_{loc}(\overline{\mathbb{R}^n_-})$ up to a subsequence and also that $\lim_{R\to+\infty} \tilde{v}_R = \tilde{v}$ in $H^2_{1,loc}(\mathbb{R}^n_-)$. Letting $R \to +\infty$ we obtain

$$\Delta \tilde{v} = \tilde{v}^{2^*-1} \qquad \text{in } \mathscr{D}'(\mathbb{R}^n_-)$$

This proves our claim and ends Step 2.3.

Step 2.4: we now conclude to prove (4.19). Let $\tilde{x}_\epsilon \in U$ be such that $\mathcal{T}_0(\tilde{x}_\epsilon) = x_\epsilon$. Then for all $\epsilon > 0$

$$\tilde{v}_\epsilon \left(\frac{\tilde{x}_\epsilon}{\mu_\epsilon} \right) = 1$$

From the properties (4.18) of the boundary chart $\mathcal{T}_0$ it follows that, for all $\epsilon > 0$

$$\frac{|\tilde{x}_\epsilon|}{\mu_\epsilon} = O\left(\frac{|x_\epsilon|}{\mu_\epsilon} \right)$$

So if $\frac{|x_\epsilon|}{\mu_\epsilon} = O(1)$ as $\epsilon \to 0$, then there exists $\tilde{x} \in \overline{\mathbb{R}^n_-}$ such that

$$\frac{\tilde{x}_\epsilon}{\mu_\epsilon} \longrightarrow \tilde{x} \qquad as\ \epsilon \to 0$$

For $R > 0$ sufficiently large we have

$$\tilde{v}_R(\tilde{x}) = \lim_{\epsilon\to 0} (\eta_R \tilde{v}_\epsilon) \left(\frac{\tilde{x}_\epsilon}{\mu_\epsilon} \right) = 1$$

and therefore

$$\tilde{v}(\tilde{x}) = \lim_{R\to+\infty} \tilde{v}_R(\tilde{y}) = 1$$

From Step 2.3, it follows that $\tilde{x} \in \mathbb{R}^n_-$. But then this implies $\tilde{v} \in C^1(\overline{\mathbb{R}^n_-})$ is a nontrivial weak solution of the equation

$$\begin{cases} \Delta \tilde{v} = \tilde{v}^{2^*-1} & \text{in } \mathbb{R}^n_- \\ \tilde{v} = 0 & \text{on } \{x_1 = 0\} \end{cases}$$

which is impossible, see Struwe's book [**18**] (Chapter III , theorem 1.3). Hence one must have that

$$\lim_{\epsilon\to 0} \frac{|x_\epsilon|}{\mu_\epsilon} = +\infty$$

This completes the proof of (4.19), and therefore Step 2. □

Step 3: We claim that

$$\lim_{\epsilon\to 0} \frac{d(x_\epsilon, \partial\Omega)}{k_\epsilon} = +\infty \tag{4.25}$$

Proof. We proceed by contradiction and assume that

$$\frac{d(x_\epsilon, \partial\Omega)}{k_\epsilon} = O(1) \qquad \text{as } \epsilon \to 0$$

Then we have that

$$\lim_{\epsilon\to 0} x_\epsilon = x_0 \in \partial\Omega$$

Step 3.1: Let $\mathcal{T}$ be a parametrisation of the boundary $\partial\Omega$ as in (4.18) around the point $p = x_0$. For all $\epsilon > 0$ let

$$\tilde{u}_\epsilon = u_\epsilon \circ \mathcal{T} \qquad \text{on } U \cap \{x_1 \leq 0\}$$

For $i, j = 1, \ldots, n$, we let $g_{ij} = (\partial_i \mathcal{T}, \partial_j \mathcal{T})$ be the metric induced by the chart $\mathcal{T}$ on the domain $U \cap \{x_1 < 0\}$, and let Δ_g denote the Laplace-Beltrami operator with respect to the metric g.

From eqn (4.4) it follows that for any $\epsilon > 0$, $\tilde{u}_\epsilon$ satisfies weakly the equation

$$\begin{cases} \Delta \tilde{u}_\epsilon + a \circ \mathcal{T}(x) \tilde{u}_\epsilon = \frac{\tilde{u}_\epsilon^{2^*(s_\epsilon)-1}}{|\mathcal{T}(x)|^{s_\epsilon}} & \text{in } U \cap \{x_1 < 0\} \\ \tilde{u}_\epsilon = 0 & \text{on } U \cap \{x_1 = 0\} \end{cases}$$

Let $z_\epsilon \in \partial\Omega$ be such that

$$|z_\epsilon - x_\epsilon| = d(x_\epsilon, \partial\Omega) \qquad \text{for } \epsilon > 0$$

And let $\tilde{x}_\epsilon$, $\tilde{z}_\epsilon \in U$ be such that

$$\mathcal{T}(\tilde{x}_\epsilon) = x_\epsilon \qquad \text{and} \qquad \mathcal{T}(\tilde{z}_\epsilon) = z_\epsilon$$

Then it follows from the properties of the boundary chart $\mathcal{T}$, that

$$\lim_{\epsilon \to 0} \tilde{x}_\epsilon = 0 = \lim_{\epsilon \to 0} \tilde{z}_\epsilon \ , \qquad (\tilde{x}_\epsilon)_1 < 0 \ \text{ and } (\tilde{z}_\epsilon)_1 = 0$$

For $\epsilon > 0$ we set

$$\tilde{v}_\epsilon = \frac{\tilde{u}_\epsilon\left(\tilde{z}_\epsilon + k_\epsilon x\right)}{\tilde{u}_\epsilon(\tilde{x}_\epsilon)} \qquad \text{for } x \in \frac{U - \tilde{z}_\epsilon}{k_\epsilon} \cap \{x_1 \leq 0\}$$

Step 3.2: For any $\eta \in C_c^\infty(\mathbb{R}^n)$, one has that $\eta \tilde{v}_\epsilon \in \mathscr{D}^{1,2}(\mathbb{R}^n_-)$ for $\epsilon > 0$ sufficiently small. Let $x \in \mathbb{R}^n_-$, then

$$\nabla\left(\eta \tilde{v}_\epsilon\right)(x) = \tilde{v}_\epsilon(x) \nabla \eta(x) + \frac{k_\epsilon}{u_\epsilon(x_\epsilon)}\ \eta(x) D_{(\tilde{z}_\epsilon + k_\epsilon x)} \mathcal{T}\left[\nabla u_\epsilon\left(\mathcal{T}(\tilde{z}_\epsilon + k_\epsilon x)\right)\right]$$

One has the inequality : For any $\theta > 0$, there exists $C(\theta) > 0$ such that for any $a, b > 0$

$$(a+b)^2 \leq C(\theta) a^2 + (1+\theta) b^2$$

With this inequality we then obtain

$$\int_{\mathbb{R}^n_-} |\nabla(\eta \tilde{v}_\epsilon)|^2 \ dx \leq C(\theta) \int_{\mathbb{R}^n_-} |\nabla \eta|^2 \tilde{v}_\epsilon^2 \ dx + (1+\theta) \frac{k_\epsilon^2}{u_\epsilon^2(x_\epsilon)} \int_{\mathbb{R}^n_-} \eta^2 \left| D_{(\tilde{z}_\epsilon + k_\epsilon x)} \mathcal{T}\left[\nabla u_\epsilon\left(\mathcal{T}(\tilde{z}_\epsilon + k_\epsilon x)\right)\right]\right|^2 \ dx$$

Since $D_0 \mathcal{T} = \mathbb{I}_{\mathbb{R}^n}$, we have as $\epsilon \to 0$

$$\int_{\mathbb{R}^n_-} |\nabla(\eta \tilde{v}_\epsilon)|^2 \ dx \leq C(\theta) \int_{\mathbb{R}^n_-} |\nabla \eta|^2 \tilde{v}_\epsilon^2 \ dx + (1+\theta)\left(1 + O(1+k_\epsilon)\right) \frac{k_\epsilon^2}{u_\epsilon^2(x_\epsilon)} \int_{\mathbb{R}^n_-} \eta^2 \left|\nabla u_\epsilon\left(\mathcal{T}(\tilde{z}_\epsilon + k_\epsilon x)\right)\right|^2 (1 + o(1)) d$$

With Hölder inequality and a change of variables this becomes

$$\int_{\mathbb{R}^n_-} |\nabla(\eta \tilde{v}_\epsilon)|^2 \ dx \leq C(\theta) \left(\frac{\mu_\epsilon}{k_\epsilon}\right)^{n-2} \|\nabla \eta\|_{L^n}^2 \left(\int_\Omega u_\epsilon^{2^*} \ dx\right)^{\frac{n-2}{n}} + (1+\theta) O(k_\epsilon) \left(\frac{\mu_\epsilon}{k_\epsilon}\right)^{n-2} \int_{\mathbb{R}^n} |\nabla u_\epsilon|^2 \ dx$$

Then by our definition (4.15) and *Sobolev inequality* (4.7) we obtain for $\epsilon > 0$ small enough

$$\int_{\mathbb{R}^n_-} |\nabla(\eta\tilde{v}_\epsilon)|^2 \, dx \le \left[C(\theta)\|\nabla\eta\|^2_{L^n} + (1+\theta)O(k_\epsilon)\right]\left(\frac{\mu_\epsilon}{|x_\epsilon|}\right)^{\frac{n-2}{2}s_\epsilon} \int_{\mathbb{R}^n} |\nabla u_\epsilon|^2 \, dx$$

$$\le \left[C(\theta)\|\nabla\eta\|^2_{L^n} + (1+\theta)O(k_\epsilon)\right]\int_{\mathbb{R}^n} |\nabla u_\epsilon|^2 \, dx \tag{4.26}$$

$$\text{since } \lim_{\epsilon\to 0}\frac{|x_\epsilon|}{\mu_\epsilon} = +\infty \text{ by eq. (4.19)}$$

Now $\|u_\epsilon\|_{H^2_{1,0}(\Omega)} = O(1)$ and $k_\epsilon \to 0$ as $\epsilon \to 0$, so for $\epsilon > 0$ small enough

$$\|\eta\tilde{v}_\epsilon\|_{\mathscr{D}^{1,2}(\mathbb{R}^n_-)} \le C_\eta$$

Where C_η is a constant depending on the function η. It then follows that there exists $v_\eta \in \mathscr{D}^{1,2}(\mathbb{R}^n_-)$ such that upto a subsequence

$$\begin{cases} \eta\tilde{v}_\epsilon \rightharpoonup \tilde{v}_\eta & \text{weakly in } \mathscr{D}^{1,2}(\mathbb{R}^n_-) \text{ as } \epsilon \to 0 \\ \eta\tilde{v}_\epsilon(x) \to \tilde{v}_\eta(x) & a.e\ x \text{ in } \mathbb{R}^n_- \text{ as } \epsilon \to 0 \end{cases}$$

Step 3.4: Let $\eta_1 \in C_c^\infty(\mathbb{R}^n)$, $0 \le \eta_1 \le 1$ be a smooth cut-off function, such that

$$\eta_1 = \begin{cases} 1 & \text{for } x \in B_0(1) \\ 0 & \text{for } x \in \mathbb{R}^n \backslash B_0(2) \end{cases}$$

For any $R > 0$ we let $\eta_R = \eta_1(x/R)$.Then with a diagonal argument we can assume that, upto a subsequence for any $\epsilon > 0$, there exists $\tilde{v}_R \in \mathscr{D}^{1,2}(\mathbb{R}^n_-)$ such that

$$\begin{cases} \eta_R\tilde{v}_\epsilon \rightharpoonup \tilde{v}_R & \text{weakly in } \mathscr{D}^{1,2}(\mathbb{R}^n_-) \text{ as } \epsilon \to 0 \\ \eta_R\tilde{v}_\epsilon \to \tilde{v}_R & a.e \text{ in } \mathbb{R}^n_- \text{ as } \epsilon \to 0 \end{cases}$$

Since $\|\nabla\eta_R\|^2_n = \|\nabla\eta_1\|^2_n$ for all $R > 0$, letting $\epsilon \to 0$ in (4.26) we obtain that

$$\int_{\mathbb{R}^n_-} |\nabla\tilde{v}_R|^2 \, dx \le C \qquad \text{for all } R > 0$$

where C is a constant independent of R. So there exists $\tilde{v} \in \mathscr{D}^{1,2}(\mathbb{R}^n_-)$ such that

$$\begin{cases} \tilde{v}_R \rightharpoonup \tilde{v} & \text{weakly in } \mathscr{D}^{1,2}(\mathbb{R}^n_-) \text{ as } R \to \infty \\ \tilde{v}_R(x) \to \tilde{v}(x) & a.e\ x \text{ in } \mathbb{R}^n_- \text{ as } R \to \infty \end{cases}$$

Step 3.5: We claim that $\tilde{v} \in C^1(\overline{\mathbb{R}^n_-})$ and it satisfies weakly the equation

$$\begin{cases} \Delta\tilde{v} = \tilde{v}^{2^*-1} & \text{in } \mathbb{R}^n_- \\ \tilde{v} = 0 & \text{on } \{x_1 = 0\} \end{cases}$$

Let

$$\tilde{g}_\epsilon = g(\tilde{z}_\epsilon + k_\epsilon x)$$

Then from equation (4.4) it follows that for any $\epsilon > 0$ and $R > 0$, $\eta_R\tilde{v}_\epsilon$ satisfies weakly the equation

$$\begin{cases} \Delta(\eta_R\tilde{v}_\epsilon) + k_\epsilon^2\left(a \circ \mathcal{T}(\tilde{z}_\epsilon + k_\epsilon x)\right)(\eta_R\tilde{v}_\epsilon) = \frac{(\eta_R\tilde{v}_\epsilon)^{2^*(s_\epsilon)-1}}{\left|\frac{\mathcal{T}(\tilde{z}_\epsilon + k_\epsilon x)}{|x_\epsilon|}\right|^{s_\epsilon}} & \text{in } B_0(R) \cap \{x_1 < 0\} \\ \eta_R\tilde{v}_\epsilon = 0 & \text{on } B_0(R) \cap \{x_1 = 0\} \end{cases} \tag{4.27}$$

From the properties of the boundary chart $\mathcal{T}$ it follows that for $\epsilon > 0$ small

$$\mathcal{T}(\tilde{z}_\epsilon + k_\epsilon x) = x_\epsilon + O_R(1)k_\epsilon \qquad \text{for } x \in B_0(R) \cap \{x_1 \leq 0\}$$

where

$$|O_R(1)| \leq C_R$$

for some $C_R > 0$. Using eq. (4.19) we obtain $\lim_{\epsilon\to 0} \frac{k_\epsilon}{|x_\epsilon|} = \lim_{\epsilon\to 0} \left(\frac{\mu_\epsilon}{|x_\epsilon|}\right)^{\frac{2-s_\epsilon}{s_\epsilon}} = 0$. So

$$\lim_{\epsilon\to 0} \left|\frac{\mathcal{T}(\tilde{z}_\epsilon + k_\epsilon x)}{|x_\epsilon|}\right|^{s_\epsilon} = 1 \qquad \text{in } C^0\left(B_0(R) \cap \{x_1 \leq 0\}\right)$$

Equation (4.27) then can be written as

$$\begin{cases} \Delta(\eta_R \tilde{v}_\epsilon) + k_\epsilon^2 \left(a \circ \mathcal{T}(\tilde{z}_\epsilon + k_\epsilon x)\right)(\eta_R \tilde{v}_\epsilon) = (1 + o(1))(\eta_R \tilde{v}_\epsilon)^{2^*(s_\epsilon)-1} & \text{in } B_0(R) \cap \{x_1 < 0\} \\ \text{with} \qquad \eta_R \tilde{v}_\epsilon = 0 & \text{on } B_0(R) \cap \{x_1 = 0\} \end{cases} \tag{4.28}$$

where $\lim_{\epsilon\to 0} o(1) = 0$ in $C^0\left(B_0(R) \cap \{x_1 \leq 0\}\right)$.

Since $0 \leq \tilde{v}_\epsilon \leq 1$, it follows from standard elliptic estimates (see for instance [**14**]) that the sequence $(\eta_R \tilde{v}_\epsilon)_{\epsilon > 0}$ is bounded in $C^{1,\alpha_0}\left(B_0(R) \cap \{x_1 \leq 0\}\right)$ for some $\alpha_0 \in (0,1)$. So by *Arzela-Ascoli's theorem* one has that $\tilde{v}_R \in C^{1,\alpha}\left(B_0(R/2) \cap \{x_1 \leq 0\}\right)$ for $0 < \alpha < \alpha_0$, and that, up to a subsequence

$$\lim_{\epsilon\to 0} \eta_R \tilde{v}_\epsilon = \tilde{v}_R \qquad \text{in } C^{1,\alpha}\left(B_0(R/4) \cap \{x_1 \leq 0\}\right)$$

for $0 < \alpha < \alpha_0$. And therefore

$$\tilde{v}_R \equiv 0 \qquad \text{on } B_0(R/4) \cap \{x_1 = 0\} \tag{4.29}$$

Letting $\epsilon \to 0$ in eqn (4.28) gives that $\tilde{v}_R$ satisfies weakly the equation

$$\begin{cases} \Delta \tilde{v}_R = \tilde{v}_R^{2^*-1} & \text{in } B_0(R/4) \cap \{x_1 \leq 0\} \\ \tilde{v}_R = 0 & \text{on } B_0(R/4) \cap \{x_1 = 0\} \end{cases} \tag{4.30}$$

Again we have that: $0 \leq \tilde{v}_R \leq 1$, then again from standard elliptic estimates and applying the *Arzela-Ascoli's theorem* it follows that $\tilde{v} \in C^1(\overline{\mathbb{R}^n_-})$ and $\lim_{R\to+\infty} \tilde{v}_R = \tilde{v}$ in $C^1_{loc}(\overline{\mathbb{R}^n_-})$ up to a subsequence. Moreover letting $R \to +\infty$ we obtain that

$$\Delta \tilde{v} = \tilde{v}^{2^*-1} \qquad \text{in } \mathscr{D}'(\mathbb{R}^n_-)$$

This proves our claim and ends Step 3.5.

Step 3.6: we know conclude Step 3. We have that

$$\tilde{v}_\epsilon\left(\frac{\tilde{x}_\epsilon - \tilde{z}_\epsilon}{k_\epsilon}\right) = 1$$

From the properties (4.18) of the boundary chart $\mathcal{T}$ it follows that, for all $\epsilon > 0$

$$\frac{|\tilde{x}_\epsilon - \tilde{z}_\epsilon|}{k_\epsilon} = O\left(\frac{|x_\epsilon - z_\epsilon|}{k_\epsilon}\right)$$

So if $\frac{d(x_\epsilon, \partial\Omega)}{k_\epsilon} = O(1)$ as $\epsilon \to 0$, then there exists $\tilde{x} \in \overline{\mathbb{R}^n_-}$ such that

$$\frac{\tilde{x}_\epsilon - \tilde{z}_\epsilon}{k_\epsilon} \longrightarrow \tilde{x} \qquad \textit{as } \epsilon \to 0$$

For $R > 0$ sufficiently large we have

$$\tilde{v}_R(\tilde{x}) = \lim_{\epsilon \to 0} (\eta_R \tilde{v}_\epsilon)\left(\frac{\tilde{x}_\epsilon - \tilde{z}_\epsilon}{k_\epsilon}\right) = 1$$

and therefore

$$\tilde{v}(\tilde{x}) = \lim_{R \to +\infty} \tilde{v}_R(\tilde{x}) = 1$$

From Step 3.5, it follows that $\tilde{x} \in \mathbb{R}^n_-$. But then this implies $\tilde{v} \in C^1(\overline{\mathbb{R}^n_-})$ is a nontrivial weak solution of the equation

$$\begin{cases} \Delta \tilde{v} = \tilde{v}^{2^*-1} & \text{in } \mathbb{R}^n_- \\ \tilde{v} = 0 & \text{on } \{x_1 = 0\} \end{cases}$$

which is a contradiction, see Struwe's book [**18**] (Chapter III , theorem 1.3). This completes the proof of (4.25) and ends Step 3. $\square$

Step 4: we are now in position to prove Theorem 4.4. Note that the preceding step yields

$$\lim_{\epsilon \to 0} \frac{d(x_\epsilon, \partial\Omega)}{k_\epsilon} = +\infty$$

Step 4.1: For any $\eta \in C_c^\infty(\mathbb{R}^n)$, one has that $\eta v_\epsilon \in H_1^2(\mathbb{R}^n)$ for $\epsilon > 0$ sufficiently small. We claim that for any $\eta \in C_c^\infty(\mathbb{R}^n)$, there exists $v_\eta \in \mathscr{D}^{1,2}(\mathbb{R}^n)$ such that upto a subsequence

$$\eta v_\epsilon \rightharpoonup v_\eta \qquad \text{weakly in } \mathscr{D}^{1,2}(\mathbb{R}^n) \qquad \text{as } \epsilon \to 0$$

Let $x \in \mathbb{R}^n$, then for $\epsilon > 0$

$$\nabla(\eta v_\epsilon)(x) = v_\epsilon \nabla \eta(x) + \mu_\epsilon^{\frac{n-2}{2}} k_\epsilon \ \eta \ \nabla u_\epsilon(x_\epsilon + k_\epsilon x)$$

One has the inequality : For any $\theta > 0$, there exists $C(\theta) > 0$ such that for any $x, y > 0$

$$(x+y)^2 \le C(\theta)x^2 + (1+\theta)y^2$$

With the help of the above inequality we then obtain

$$\int_{\mathbb{R}^n} |\nabla(\eta v_\epsilon)|^2 \ dx \le C(\theta) \int_{\mathbb{R}^n} |\nabla \eta|^2 v_\epsilon^2 \ dx + (1+\theta)\mu_\epsilon^{n-2} k_\epsilon^2 \int_{\mathbb{R}^n} \eta^2 |\nabla u_\epsilon(x_\epsilon + k_\epsilon x)|^2 \ dx$$

With Hölder inequality and a change of variables this becomes

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla(\eta v_\epsilon)|^2 \ dx \le & \left(\frac{\mu_\epsilon}{k_\epsilon}\right)^{n-2} C(\theta) \|\nabla \eta\|_{L^n}^2 \left(\int_{\mathbb{R}^n} u_\epsilon^{2^*} \ dx\right)^{\frac{n-2}{n}} \\ & + (1+\theta)\left(\frac{\mu_\epsilon}{k_\epsilon}\right)^{n-2} \int_{\mathbb{R}^n} \left(\eta\left(\frac{x - x_\epsilon}{k_\epsilon}\right)\right)^2 |\nabla u_\epsilon|^2 \ dx \end{aligned} \tag{4.31}$$

By the *Sobolev inequality* (4.7) and our definition of k_ϵ, we obtain for $\epsilon > 0$ small enough

$$\int_{\mathbb{R}^n} |\nabla (\eta v_\epsilon)|^2 \, dx \le \left[C(\theta) \|\nabla \eta\|_{L^n}^2 + (1+\theta)\sup \eta^2\right] \left(\frac{\mu_\epsilon}{|x_\epsilon|}\right)^{\frac{n-2}{2}s_\epsilon} \int_{\mathbb{R}^n} |\nabla u_\epsilon|^2 \, dx$$

$$\le \left[C(\theta) \|\nabla \eta\|_{L^n}^2 + (1+\theta)\sup \eta^2\right] \int_{\mathbb{R}^n} |\nabla u_\epsilon|^2 \, dx$$

$$\text{since } \lim_{\epsilon \to 0} \frac{|x_\epsilon|}{\mu_\epsilon} = +\infty \quad \text{by eq. (4.19)}$$

Now $\|u_\epsilon\|_{H^2_{1,0}(\Omega)} = O(1)$ and $k_\epsilon \to 0$ as $\epsilon \to 0$, so for $\epsilon > 0$ small enough

$$\|\eta v_\epsilon\|_{\mathscr{D}^{1,2}(\mathbb{R}^n)} \le C_\eta$$

Where C_η is a constant depending on the function η. It then follows that there exists $v_\eta \in \mathscr{D}^{1,2}(\mathbb{R}^n)$ such that upto a subsequence

$$\left\{ \begin{array}{lll} \eta v_\epsilon \rightharpoonup v_\eta & \text{weakly in } \mathscr{D}^{1,2}(\mathbb{R}^n) & \text{as } \epsilon \to 0 \\ \eta v_\epsilon(x) \to v_\eta(x) & a.e\ x \ \text{ in } \mathbb{R}^n & \text{as } \epsilon \to 0 \end{array} \right. \tag{4.32}$$

This proves the claim and ends Step 4.1.

Step 4.2: We claim that there exists $v \in \mathscr{D}^{1,2}(\mathbb{R}^n)$ such that for any $\eta \in C_c^\infty(\mathbb{R}^n)$ we have

$$v_\eta = \eta v$$

Let $\eta_1 \in C_c^\infty(\mathbb{R}^n)$, $0 \le \eta_1 \le 1$ be a smooth cut-off function, such that

$$\eta_1 = \left\{ \begin{array}{ll} 1 & \text{for} \quad x \in B_0(1) \\ 0 & \text{for} \quad x \in \mathbb{R}^n \backslash B_0(2) \end{array} \right.$$

For any $R > 0$ we let $\eta_R = \eta_1(x/R)$. Then with a diagonal argument we can assume that, upto a subsequence for any $\epsilon > 0$, there exists $v_R \in \mathscr{D}^{1,2}(\mathbb{R}^n)$ such that

$$\left\{ \begin{array}{lll} \eta_R v_\epsilon \rightharpoonup v_R & \text{weakly in } \mathscr{D}^{1,2}(\mathbb{R}^n) & \text{as } \epsilon \to 0 \\ \eta_R v_\epsilon \to v_R & a.e \ \text{ in } \mathbb{R}^n & \text{as } \epsilon \to 0 \end{array} \right.$$

Since $\|\nabla \eta_R\|_n^2 = \|\nabla \eta_1\|_n^2$ for all $R > 0$, letting $\epsilon \to 0$ in (4.31) we obtain that

$$\int_{\mathbb{R}^n} |\nabla v_R|^2 \, dx \le C \qquad \text{for all } R > 0$$

where C is a constant independent of R. So there exists $v \in \mathscr{D}^{1,2}(\mathbb{R}^n)$ such that

$$\left\{ \begin{array}{lll} v_R \rightharpoonup v & \text{weakly in } \mathscr{D}^{1,2}(\mathbb{R}^n) & \text{as } R \to \infty \\ v_R(x) \to v(x) & a.e\ x \ \text{ in } \mathbb{R}^n & \text{as } R \to \infty \end{array} \right.$$

And therefore for any $\eta \in C_c^\infty(\mathbb{R}^n)$

$$v_\eta = \eta v$$

This ends Step 4.2.

Step 4.3: We claim that $v \in C^1(\mathbb{R}^n)$, $v \neq 0$ and it satisfies weakly the equation

$$\Delta v = v^{2^*-1} \qquad \text{in } \mathbb{R}^n$$

We prove the claim. Using eqn (4.4) it follows that for any $\epsilon > 0$ and $R > 0$, $\eta_R v_\epsilon$ satisfies the equation

$$\Delta(\eta_R v_\epsilon) + k_\epsilon^2 a(x_\epsilon + k_\epsilon x)(\eta_R v_\epsilon) = \frac{(\eta_R v_\epsilon)^{2^*(s_\epsilon)-1}}{\left|\frac{x_\epsilon}{|x_\epsilon|} + \frac{k_\epsilon}{|x_\epsilon|}x\right|^{s_\epsilon}} \qquad \text{in } \mathscr{D}'(B_0(R)) \tag{4.33}$$

From eq. (4.19) we obtain $\lim_{\epsilon\to 0} \frac{k_\epsilon}{|x_\epsilon|} = \lim_{\epsilon\to 0}\left(\frac{\mu_\epsilon}{|x_\epsilon|}\right)^{\frac{2-s_\epsilon}{s_\epsilon}} = 0$. So we have

$$\lim_{\epsilon\to 0}\left|\frac{x_\epsilon}{|x_\epsilon|} + \frac{k_\epsilon}{|x_\epsilon|}x\right|^{s_\epsilon} = 1 \qquad \text{in } C^0(B_0(R))$$

Then equation (4.33) then can be written as

$$\Delta(\eta_R v_\epsilon) + k_\epsilon^2 a(x_\epsilon + k_\epsilon x)(\eta_R v_\epsilon) = (1 + o(1))(\eta_R v_\epsilon)^{2^*(s_\epsilon)-1} \qquad \text{in } \mathscr{D}'(B_0(R)) \tag{4.34}$$

where $\lim_{\epsilon\to 0} o(1) = 0$ in $C^0(B_0(R))$.

Since $0 \le v_\epsilon \le 1$, it follows from standard elliptic estimates (see for instance [**14**]) that $v_R \in C^1(B_0(R))$, and up to a subsequence

$$\lim_{\epsilon\to 0} \eta_R v_\epsilon = v_R \qquad \text{in } C^1_{loc}(B_0(R))$$

Letting $\epsilon \to 0$ in eqn (4.34) gives that v_R satisfies the equation

$$\Delta v_R = v_R^{2^*-1} \qquad \text{in } \mathscr{D}'(B_0(R))$$

Further as for any $\epsilon > 0$ and $R > 0$, $\eta_R v_\epsilon(0) = 1$, therefore $v_R(0) = 1$ for all $R > 0$. Moreover $\max_{x\in B_0(R)} v_R(x) = 1$.

Again we have that: $0 \le v_R \le 1$ since $\eta_R v_\epsilon \to v_R$ *a.e* in $\mathbb{R}^n$ as $\epsilon \to 0$. Then again from standard elliptic estimates it follows that $v \in C^1(\mathbb{R}^n)$ and $\lim_{R\to+\infty} v_R = v$ in $C^1_{loc}(\mathbb{R}^n)$ up to a subsequence. Letting $R \to +\infty$ we obtain that

$$\Delta v = v^{2^*-1} \qquad \text{in } \mathscr{D}'(\mathbb{R}^n)$$

Further we have that $\max_{x\in\mathbb{R}^n} v(x) = v(0) = 1$. By Cafferelli, Gidas and Spruck classification of nonegative $\mathscr{D}^{1,2}(\mathbb{R}^n)$ solutions of the equation $\Delta v = v^{2^*-1}$ we then have :

$$v(x) = \left(\frac{1}{1 + \frac{|x|^2}{n(n-2)}}\right)^{\frac{n-2}{2}} \qquad \text{for all } x \in \mathbb{R}^n$$

Moreover,

$$v_\epsilon \longrightarrow v \qquad in\ C^1_{loc}(\mathbb{R}^n) \qquad as\ \epsilon \to 0$$

This proves our claim and ends Step 4.3.

Step 4.4: Coming back to equation (4.31) we have for $R > 0$

$$\int_{\mathbb{R}^n} |\nabla(\eta_R v_\epsilon)|^2 \, dx \leq C(\theta) \|\nabla \eta_R\|_{L^n}^2 \left(\int_{B_0(2R)\setminus B_0(R)} (\eta_{2R} v_\epsilon)^{2^*} \, dx \right)^{\frac{n-2}{n}}$$
$$+ (1+\theta) \left(\frac{\mu_\epsilon}{k_\epsilon}\right)^{n-2} \int_{\Omega \cap B_{x_\epsilon}(2Rk_\epsilon)} \left(\eta_R\left(\frac{x - x_\epsilon}{k_\epsilon}\right)\right)^2 |\nabla u_\epsilon|^2 \, dx$$

$$\leq C(\theta) \left(\int_{B_0(2R)\setminus B_0(R)} (\eta_{2R} v_\epsilon)^{2^*} \, dx \right)^{\frac{n-2}{n}} + (1+\theta) \left(\frac{\mu_\epsilon}{k_\epsilon}\right)^{n-2} \int_\Omega |\nabla u_\epsilon|^2 \, dx \tag{4.35}$$

Now $u_\epsilon \rightharpoonup 0$ weakly in $H_{1,0}^2(\Omega)$ as $\epsilon \to 0$, where for each $\epsilon > 0$, u_ϵ satisfies (4.4) and (4.5). So we have

$$\int_\Omega |\nabla u_\epsilon|^2 \, dx = \int_\Omega \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \, dx + o(1) \leq \mu_{s,a}(\Omega)^{\frac{2^*(s_\epsilon)}{2^*(\epsilon)-2}} + o(1) \qquad \text{as } \epsilon \to 0$$

Letting $\epsilon \to 0$ we obtain, using Step 4.2 and proposition 4.3.1, that for $R > 0$

$$\int_{\mathbb{R}^n} |\nabla v_R|^2 \, dx \leq C(\theta) \left(\int_{B_0(2R)\setminus B_0(R)} v^{2^*} \, dx \right)^{\frac{n-2}{n}} + (1+\theta) \left(\limsup_{\epsilon\to 0} \left(\frac{\mu_\epsilon}{|x_\epsilon|}\right)^{s_\epsilon} \right)^{\frac{n-2}{2}} \mu_a(\Omega)^{\frac{2^*}{2^*-2}}$$

And then letting $R \to +\infty$ gives us

$$\int_{\mathbb{R}^n} |\nabla v|^2 \, dx \leq (1+\theta) \left(\limsup_{\epsilon\to 0} \left(\frac{\mu_\epsilon}{|x_\epsilon|}\right)^{s_\epsilon} \right)^{\frac{n-2}{2}} \mu_a(\Omega)^{\frac{2^*}{2^*-2}}$$

Since $\theta > 0$ is arbitrary, this implies that

$$\int_{\mathbb{R}^n} |\nabla v|^2 \, dx \leq \left(\lim_{\epsilon\to 0} \left(\frac{\mu_\epsilon}{|x_\epsilon|}\right)^{s_\epsilon} \right)^{\frac{n-2}{2}} \mu_a(\Omega)^{\frac{2^*}{2^*-2}} \leq \left(\limsup_{\epsilon\to 0} \left(\frac{\mu_\epsilon}{|x_\epsilon|}\right)^{s_\epsilon} \right)^{\frac{n-2}{2}} \mu_a(\Omega)^{\frac{2^*}{2^*-2}} \tag{4.36}$$

From eq. (4.19) we have $\limsup_{\epsilon\to 0} \left(\frac{\mu_\epsilon}{|x_\epsilon|}\right)^{s_\epsilon} \leq 1$, and since $\mu_a(\Omega) \leq \frac{1}{K(n,0)}$ (see for instance Aubin [**2**])

$$\int_{\mathbb{R}^n} |\nabla v|^2 \, dx \leq \mu_a(\Omega)^{\frac{2^*}{2^*-2}} \leq \left(\frac{1}{K(n,0)}\right)^{\frac{2^*}{2^*-2}}$$

Now

$$\Delta v = v^{2^*-1} \qquad \text{in } \mathscr{D}'(\mathbb{R}^n_-)$$

Then by *Sobolev inequality* (4.7)

$$\int_{\mathbb{R}^n} |\nabla v|^2 \, dx \geq \left(\frac{1}{K(n,0)}\right)^{\frac{2^*}{2^*-2}}$$

Hence we have

$$\int_{\mathbb{R}^n} |\nabla v|^2 \ dx = \left(\frac{1}{K(n,0)} \right)^{\frac{2^*}{2^*-2}}$$

Then (4.36) implies that

$$\limsup_{\epsilon \to 0} \left(\frac{\mu_\epsilon}{|x_\epsilon|} \right)^{s_\epsilon} \geq 1$$

and we have

$$\lim_{\epsilon \to 0} \left(\frac{\mu_\epsilon}{|x_\epsilon|} \right)^{s_\epsilon} = 1 \ , \quad \lim_{\epsilon \to 0} \frac{k_\epsilon}{\mu_\epsilon} = 1 \tag{4.37}$$

This ends Step 4.4 and completes the proof of Theorem 4.4. □

As a consequence of Theorem 4.4, we get the following concentration of energy:

Proposition 4.4.1. *Under the hypothesis of Theorem* 4.4 *one further has that*

$$\lim_{R \to +\infty} \lim_{\epsilon \to 0} \int_{\Omega \setminus B_{x_\epsilon}(Rk_\epsilon)} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \ dx = 0$$

PROOF. We obtain by change of variables

$$\begin{aligned}
\int_{\Omega \setminus B_{x_\epsilon}(Rk_\epsilon)} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \ dx &= \int_{\Omega} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \ dx - \int_{B_{x_\epsilon}(Rk_\epsilon)} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \ dx \\
&= \int_{\Omega} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \ dx - \frac{k_\epsilon^n}{\mu_\epsilon^{n-s_\epsilon}} \int_{B_0(R)} \frac{|v_\epsilon(x)|^{2^*(s_\epsilon)}}{|x_\epsilon + k_\epsilon x|^{s_\epsilon}} \ dx \\
&= \int_{\Omega} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \ dx - \left(\frac{|x_\epsilon|^{s_\epsilon}}{\mu_\epsilon^{s_\epsilon}} \right)^{\frac{n-2}{2}} \int_{B_0(R)} \frac{|v_\epsilon(x)|^{2^*(s_\epsilon)}}{\left| \frac{x_\epsilon}{|x_\epsilon|} + \frac{k_\epsilon}{|x_\epsilon|} x \right|^{s_\epsilon}} \ dx
\end{aligned}$$

Letting $\epsilon \to 0$ and then $R \to +\infty$ one obtains the proposition using Theorem 4.4. □

4.5. Refined Blowup Analysis I

In this section we obtain pointwise bounds on the blowup sequence $(u_\epsilon)_{\epsilon > 0}$ that will be used in next section to get the optimal bound.

Theorem 4.5. *With the same hypothesis as in Theorem* 4.4*, we have that there exists a constant* $C > 0$ *such that for* $\epsilon > 0$

$$|x - x_\epsilon|^{\frac{n-2}{2}} u_\epsilon(x) + \frac{|x - x_\epsilon|^{\frac{n}{2}}}{d(x, \partial\Omega)} u_\epsilon(x) \leq C \qquad \textit{for all } x \in \Omega.$$

Moreover,

$$\lim_{R \to +\infty} \lim_{\epsilon \to 0} \sup_{x \in \Omega \setminus B_{x_\epsilon}(Rk_\epsilon)} |x - x_\epsilon|^{\frac{n-2}{2}} u_\epsilon(x) = 0$$

The proof of Theorem 4.5 goes through the proof of the three propositions below.

Proposition 4.5.1. *With the same hypothesis as in Theorem* 4.4, *we have that there exists a constant* $C>0$ *such that for* $\epsilon>0$

$$|x-x_\epsilon|^{\frac{n-2}{2}}u_\epsilon(x)\le C \qquad \text{for all } x\in\Omega$$

PROOF. Suppose on the contrary

$$\sup_{x\in\Omega}\left(|x-x_\epsilon|^{\frac{n-2}{2}}u_\epsilon(x)\right)\longrightarrow+\infty \qquad \text{as } \epsilon\to0$$

Let $y_\epsilon\in\Omega$ be such that

$$|y_\epsilon-x_\epsilon|^{\frac{n-2}{2}}u_\epsilon(y_\epsilon)=\sup_{x\in\Omega}\left(|x-x_\epsilon|^{\frac{n-2}{2}}u_\epsilon(x)\right)$$

Then

$$|y_\epsilon-x_\epsilon|^{\frac{n-2}{2}}u_\epsilon(y_\epsilon)\longrightarrow+\infty \qquad \text{as } \epsilon\to0 \tag{4.38}$$

We let

$$\lambda_\epsilon^{-\frac{n-2}{2}}=u_\epsilon(y_\epsilon)$$

then $\mu_\epsilon\le\lambda_\epsilon$ and (4.38) becomes

$$\lim_{\epsilon\to0}\frac{|y_\epsilon-x_\epsilon|}{\lambda_\epsilon}=+\infty$$

and so we have that

$$\lim_{\epsilon\to0}\lambda_\epsilon=0$$

Step 1. As our first step we show that

$$\lim_{\epsilon\to0}\frac{|y_\epsilon|}{\lambda_\epsilon}=+\infty \tag{4.39}$$

PROOF. Suppose on the contrary

$$\frac{|y_\epsilon|}{\lambda_\epsilon}=O(1) \qquad \text{as } \epsilon\to0$$

Then this implies that

$$\lim_{\epsilon\to+\infty}|y_\epsilon|=0$$

Let $\mathcal{T}_0:U\to V$ be a parametrisation of the boundary as in (4.18) around the point $p=0$. For all $\epsilon>0$, we let

$$\tilde{w}_\epsilon(x)=\frac{u_\epsilon\circ\mathcal{T}_0(\lambda_\epsilon x)}{u_\epsilon(y_\epsilon)} \qquad \text{for } x\in\frac{U}{\lambda_\epsilon}\cap\{x_1\le0\}$$

Step 1.1: For any $\eta\in C_c^\infty(\mathbb{R}^n)$, one has that $\eta\tilde{w}_\epsilon\in H^2_{1,0}(\mathbb{R}^n_-)$ for $\epsilon>0$ sufficiently small. Let $x\in\mathbb{R}^n_-$, then

$$\nabla\left(\eta\tilde{w}_\epsilon\right)(x)=\tilde{w}_\epsilon(x)\nabla\eta(x)+\frac{\lambda_\epsilon}{u_\epsilon(y_\epsilon)}\,\eta(x)D_{(\lambda_\epsilon x)}\mathcal{T}_0\left[\nabla u_\epsilon\left(\mathcal{T}_0(\lambda_\epsilon x)\right)\right]$$

One has the inequality : For any $\theta>0$, there exists $C(\theta)>0$ such that for any $a,b>0$

$$(a+b)^2\le C(\theta)a^2+(1+\theta)b^2$$

With this inequality we then obtain

$$\int_{\mathbb{R}^n_-} |\nabla(\eta\tilde{w}_\epsilon)|^2 \ dx \le C(\theta)\int_{\mathbb{R}^n_-} |\nabla\eta|^2\tilde{w}_\epsilon^2 \ dx + (1+\theta)\frac{\lambda_\epsilon^2}{u_\epsilon^2(y_\epsilon)}\int_{\mathbb{R}^n_-} \eta^2 \left|D_{(\lambda_\epsilon x)}\mathcal{T}_0\left[\nabla u_\epsilon\left(\mathcal{T}_0(\lambda_\epsilon x)\right)\right]\right|^2 \ dx$$

Since $D_0\mathcal{T}_0 = \mathbb{I}_{\mathbb{R}^n}$ we have for $\epsilon > 0$ sufficiently small

$$\int_{\mathbb{R}^n_-} |\nabla(\eta\tilde{w}_\epsilon)|^2 \ dx \le C(\theta)\int_{\mathbb{R}^n_-} |\nabla\eta|^2\tilde{w}_\epsilon^2 \ dx + (1+\theta)\left(1+O(\lambda_\epsilon)\right)\frac{\lambda_\epsilon^2}{u_\epsilon^2(y_\epsilon)}\int_{\Omega} \eta^2 \left|\nabla u_\epsilon\left(\mathcal{T}_0(\lambda_\epsilon x)\right)\right|^2 \ dx$$

With Hölder inequality and a change of variables this becomes

(4.40)

$$\int_{\mathbb{R}^n_-} |\nabla(\eta\tilde{w}_\epsilon)|^2 \ dx \le C(\theta)\,\|\nabla\eta\|_{L^n}^2\left(\int_{\Omega} u_\epsilon^{2^*} \ dx\right)^{\frac{n-2}{n}} + (1+\theta)\left(1+O(\lambda_\epsilon)\right)\int_{\mathbb{R}^n} |\nabla u_\epsilon|^2 \ dx$$

Now since $\|u_\epsilon\|_{H^2_{1,0}(\Omega)} = O(1)$ and $\lambda_\epsilon \to 0$ as $\epsilon \to 0$, so for $\epsilon > 0$ small enough

$$\|\eta\tilde{w}_\epsilon\|_{\mathscr{D}^{1,2}(\mathbb{R}^n_-)} \le C_\eta$$

Where C_η is a constant depending on the function η. It then follows that there exists $w_\eta \in \mathscr{D}^{1,2}(\mathbb{R}^n_-)$ such that upto a subsequence

$$\left\{\begin{array}{ll} \eta\tilde{w}_\epsilon \rightharpoonup \tilde{w}_\eta & \text{weakly in } \mathscr{D}^{1,2}(\mathbb{R}^n_-) \ \text{ as } \epsilon \to 0 \\ \eta\tilde{w}_\epsilon(x) \to \tilde{w}_\eta(x) & a.e\ x \ \text{ in } \mathbb{R}^n_- \ \text{ as } \epsilon \to 0 \end{array}\right. \tag{4.41}$$

Step 1.2: Let $\eta_1 \in C_c^\infty(\mathbb{R}^n)$, $0 \le \eta_1 \le 1$ be a smooth cut-off function, such that

$$\eta_1 = \left\{\begin{array}{ll} 1 & \text{for} \quad x \in B_0(1) \\ 0 & \text{for} \quad x \in \mathbb{R}^n \backslash B_0(2) \end{array}\right.$$

For any $R > 0$ we let $\eta_R = \eta_1(x/R)$. Then with a diagonal argument we can assume that, upto a subsequence for any $R > 0$, there exists $\tilde{w}_R \in \mathscr{D}^{1,2}(\mathbb{R}^n_-)$ such that

$$\left\{\begin{array}{ll} \eta_R\tilde{w}_\epsilon \rightharpoonup \tilde{w}_R & \text{weakly in } \mathscr{D}^{1,2}(\mathbb{R}^n_-) \ \text{ as } \epsilon \to 0 \\ \eta_R\tilde{w}_\epsilon(x) \to \tilde{w}_R(x) & a.e\ x \ \text{ in } \mathbb{R}^n_- \ \text{ as } \epsilon \to 0 \end{array}\right.$$

Since $\|\nabla\eta_R\|_n^2 = \|\nabla\eta_1\|_n^2$ for all $R > 0$, letting $\epsilon \to 0$ in (4.40) we obtain that

$$\int_{\mathbb{R}^n_-} |\nabla\tilde{w}_R|^2\, dx \le C \qquad \text{for all } R > 0$$

where C is a constant independent of R. So there exists $\tilde{w} \in \mathscr{D}^{1,2}(\mathbb{R}^n_-)$ such that

$$\left\{\begin{array}{ll} \tilde{w}_R \rightharpoonup \tilde{w} & \text{weakly in } \mathscr{D}^{1,2}(\mathbb{R}^n_-) \ \text{ as } R \to \infty \\ \tilde{w}_R(x) \to \tilde{w}(x) & a.e\ x \ \text{ in } \mathbb{R}^n_- \ \text{ as } R \to \infty \end{array}\right.$$

Step 1.3: We claim that $\tilde{w} \in C^1(\overline{\mathbb{R}^n_-})$ and it satisfies weakly the equation

$$\left\{\begin{array}{ll} \Delta\tilde{w} = \tilde{w}^{2^*-1} & \text{in } \mathbb{R}^n_- \\ \tilde{w} = 0 & \text{on } \{x_1 = 0\} \end{array}\right.$$

For $i, j = 1, \ldots, n$, we let $g_{ij} = (\partial_i\mathcal{T}_0, \partial_j\mathcal{T}_0)$, the metric induced by the chart $\mathcal{T}_0$ on the domain $U \cap \{x_1 < 0\}$ and let Δ_g denote the Laplace-Beltrami operator with respect to the metric g. We let

$$\tilde{g}_\epsilon = g\left(\lambda_\epsilon x\right)$$

From eqn (4.4) it follows that for any $\epsilon > 0$ and $R > 0$, $\eta_R \tilde{w}_\epsilon$ satisfies weakly the equation

$$\left\{ \begin{array}{ll} \Delta_{\tilde{g}_\epsilon} (\eta_R \tilde{w}_\epsilon) + \lambda_\epsilon^2 (a \circ \mathcal{T}_0(\lambda_\epsilon x)) (\eta_R \tilde{w}_\epsilon) = \frac{(\eta_R \tilde{w}_\epsilon)^{2^*(s_\epsilon)-1}}{\left| \frac{\mathcal{T}_0(\lambda_\epsilon x)}{\lambda_\epsilon} \right|^{s_\epsilon}} & \text{in } B_0(R) \cap \{x_1 < 0\} \\ \eta_R \tilde{w}_\epsilon = 0 & \text{on } B_0(R) \cap \{x_1 = 0\} \end{array} \right. \tag{4.42}$$

For $R > 0$ and $\epsilon > 0$ we have

$$|\mathcal{T}_0(\lambda_\epsilon x) - x_\epsilon|^{\frac{n-2}{2}} \eta_R \tilde{w}_\epsilon(x) \leq |y_\epsilon - x_\epsilon|^{\frac{n-2}{2}} \lambda_\epsilon^{\frac{n-2}{2}} u_\epsilon(y_\epsilon),$$

$$\left(\frac{|\mathcal{T}_0(\lambda_\epsilon x) - x_\epsilon|}{|y_\epsilon - x_\epsilon|} \right)^{\frac{n-2}{2}} \eta_R \tilde{w}_\epsilon(x) \leq 1,$$

It follows from the properties of the map $\mathcal{T}_0$, that for $\epsilon > 0$ sufficiently small

$$\mathcal{T}_0(\lambda_\epsilon x) = y_\epsilon + O_R(1)\lambda_\epsilon \qquad \text{for all } x \in B_0(R) \cap \{x_1 \leq 0\}$$

where

$$|O_R(1)| \leq C_R$$

for some $C_R > 0$ depending only on R. Then since $\lim_{\epsilon \to 0} \frac{|y_\epsilon - x_\epsilon|}{\lambda_\epsilon} = +\infty$, we obtain

$$\lim_{\epsilon \to 0} \frac{|\mathcal{T}_0(\lambda_\epsilon x) - x_\epsilon|}{|y_\epsilon - x_\epsilon|} = \lim_{\epsilon \to 0} \frac{|y_\epsilon - x_\epsilon + O_R(1)\lambda_\epsilon|}{|y_\epsilon - x_\epsilon|} = 1 \qquad \text{for all } x \in B_0(R) \cap \{x_1 \leq 0\}$$

It then follows that for $\epsilon > 0$ sufficiently small

$$\eta_R \tilde{w}_\epsilon(x) \leq 2 \qquad \text{for all } x \in B_0(R) \cap \{x_1 \leq 0\}$$

Now, from the properties of the boundary chart $\mathcal{T}_0$, it follows that for any $p > 1$ there exists a constant C_p such that

$$\int_{B_0(R) \cap \{x_1 < 0\}} \left[\frac{(\eta_R \tilde{w}_\epsilon)^{2^*(s_\epsilon)-1}}{\left| \frac{\mathcal{T}_0(\lambda_\epsilon x)}{\lambda_\epsilon} \right|^{s_\epsilon}} \right]^p dx \leq C_p \int_{B_0(R) \cap \{x_1 < 0\}} \frac{1}{|x|^{s_\epsilon p}} \, dx$$

So the right hand side of equation (4.42) is uniformly bounded in L^p for some $p > n$. Then from standard elliptic estimates (see for instance [**14**]) it follows that the sequence $(\eta_R \tilde{w}_\epsilon)_{\epsilon > 0}$ is bounded in $C^{1,\alpha_0}(B_0(R) \cap \{x_1 \leq 0\})$ for some $\alpha_0 \in (0,1)$. So by *Arzela-Ascoli's theorem* one has that there exists $\tilde{w}_R \in C^{1,\alpha}(B_0(R/2) \cap \{x_1 \leq 0\})$ for $0 < \alpha < \alpha_0$, and that, up to a subsequence

$$\lim_{\epsilon \to 0} \eta_R \tilde{w}_\epsilon = \tilde{w}_R \qquad \text{in } C^{1,\alpha}(B_0(R/4) \cap \{x_1 \leq 0\})$$

for $0 < \alpha < \alpha_0$. And therefore

$$\tilde{w}_R \equiv 0 \qquad \text{on } B_0(R/4) \cap \{x_1 = 0\} \tag{4.43}$$

Letting $\epsilon \to 0$ in eqn (4.42) gives that $\tilde{w}_R$ satisfies weakly the equation

$$\left\{ \begin{array}{ll} \Delta \tilde{w}_R = \tilde{w}_R^{2^*-1} & \text{in } B_0(R/4) \cap \{x_1 \leq 0\} \\ \tilde{w}_R = 0 & \text{on } B_0(R/4) \cap \{x_1 = 0\} \end{array} \right. \tag{4.44}$$

We have that $0 \le \tilde{w}_R \le 2$, then again from standard elliptic estimates and applying the *Arzela-Ascoli's theorem* it follows that $\tilde{w} \in C^1(\overline{\mathbb{R}^n_-})$ and $\lim_{R\to+\infty} \tilde{w}_R = \tilde{w}$ in $C^1_{loc}(\overline{\mathbb{R}^n_-})$ up to a subsequence. Moreover letting $R \to +\infty$ we obtain that

$$\begin{cases} \Delta\tilde{w} = \tilde{w}^{2^*-1} & \text{in } \mathbb{R}^n_- \\ \tilde{w} = 0 & \text{on } \{x_1 = 0\} \end{cases}$$

This proves our claim and ends Step 1.3.

Step 1.4: Let $\tilde{y}_\epsilon \in U$ be such that $\mathcal{T}_0(\tilde{y}_\epsilon) = y_\epsilon$. Then for all $\epsilon > 0$

$$\tilde{w}_\epsilon \left(\frac{\tilde{y}_\epsilon}{\lambda_\epsilon} \right) = 1$$

From the properties of the boundary chart $\mathcal{T}_0$ it follows that, for all $\epsilon > 0$

$$\frac{|\tilde{y}_\epsilon|}{\lambda_\epsilon} = O\left(\frac{|y_\epsilon|}{\lambda_\epsilon} \right)$$

So if $\dfrac{|y_\epsilon|}{\lambda_\epsilon} = O(1)$ as $\epsilon \to 0$, then there exists $\tilde{y}_0 \in \overline{\mathbb{R}^n_-}$ such that

$$\frac{\tilde{y}_\epsilon}{\lambda_\epsilon} \longrightarrow \tilde{y}_0 \qquad \textit{as } \epsilon \to 0$$

For $R > 0$ sufficiently large we have

$$\tilde{w}_R(\tilde{y}) = \lim_{\epsilon\to 0} (\eta_R \tilde{w}_\epsilon) \left(\frac{\tilde{y}_\epsilon}{\mu_\epsilon} \right) = 1$$

and therefore

$$\tilde{w}(\tilde{y}_0) = \lim_{R\to+\infty} \tilde{w}_R(\tilde{y}_0) = 1$$

From (4.43), it follows that $\tilde{y}_0 \in \mathbb{R}^n_-$. But then this implies $\tilde{w} \in C^1(\overline{\mathbb{R}^n_-})$ is a nontrivial weak solution of the equation

$$\begin{cases} \Delta\tilde{w} = \tilde{w}^{2^*-1} & \text{in } \mathbb{R}^n_- \\ \tilde{w} = 0 & \text{on } \{x_1 = 0\} \end{cases}$$

which is impossible, see Struwe's book [**18**] (Chapter III , theorem 1.3). This ends Step 1.4, and therefore proves (4.39) and ends Step 1. □

We let

$$l_\epsilon = |y_\epsilon|^{s_\epsilon/2} \lambda_\epsilon^{\frac{2-s_\epsilon}{2}} \qquad \text{for } \epsilon > 0$$

Then

$$\lim_{\epsilon\to 0} l_\epsilon = 0$$

Step 2: We claim that

$$\frac{|y_\epsilon|^{s_\epsilon}}{\lambda_\epsilon^{s_\epsilon}} = O(1) \qquad \text{as } \epsilon \to 0 \tag{4.45}$$

PROOF. We proceed by contradiction. Suppose if

$$\lim_{\epsilon\to 0} \frac{\lambda_\epsilon^{s_\epsilon}}{|y_\epsilon|^{s_\epsilon}} = 0$$

Now

$$\frac{|x_\epsilon|^{s_\epsilon}}{|y_\epsilon|^{s_\epsilon}} = \frac{\lambda_\epsilon^{s_\epsilon}}{|y_\epsilon|^{s_\epsilon}} \frac{|x_\epsilon|^{s_\epsilon}}{\lambda_\epsilon^{s_\epsilon}} \leq \frac{\lambda_\epsilon^{s_\epsilon}}{|y_\epsilon|^{s_\epsilon}} \frac{|x_\epsilon|^{s_\epsilon}}{\mu_\epsilon^{s_\epsilon}}$$

Since $\lim\limits_{\epsilon\to 0} \frac{|x_\epsilon|^{s_\epsilon}}{\mu_\epsilon^{s_\epsilon}} = 1$ as shown in (4.17), it follows that one must have

$$\lim_{\epsilon\to 0} \frac{|x_\epsilon|^{s_\epsilon}}{|y_\epsilon|^{s_\epsilon}} = 0$$

And in particular $\lim\limits_{\epsilon\to 0} \frac{|x_\epsilon|}{|y_\epsilon|} = 0$.

We can have two cases:

Case 2.1: We assume that, upto a subsequence, there exists $\rho > 0$ such that

$$\frac{d(y_\epsilon, \partial\Omega)}{l_\epsilon} \geq 3\rho$$

For any $\epsilon > 0$ we let

$$w_\epsilon(x) = \lambda_\epsilon^{\frac{n-2}{2}} u_\epsilon(y_\epsilon + l_\epsilon x) \qquad \text{for } x \in B_0(2\rho)$$

This is well defined since $B_{y_\epsilon}(2l_\epsilon\rho) \subset \Omega$. Using eqn (4.4) it follows that for $\epsilon > 0$, w_ϵ satisfies the equation

$$\Delta w_\epsilon + l_\epsilon^2 a\left(y_\epsilon + l_\epsilon x\right) w_\epsilon = \frac{w_\epsilon^{2^*(s_\epsilon)-1}}{\left|\frac{y_\epsilon}{|y_\epsilon|} + \frac{l_\epsilon}{|y_\epsilon|}x\right|^{s_\epsilon}} \qquad \text{in } \mathscr{D}'(B_0(2\rho)) \tag{4.46}$$

We have

$$|l_\epsilon x + y_\epsilon - x_\epsilon|^{\frac{n-2}{2}} w_\epsilon(x) \leq |y_\epsilon - x_\epsilon|^{\frac{n-2}{2}} \lambda_\epsilon^{\frac{n-2}{2}} u_\epsilon(y_\epsilon) \qquad \text{for } \epsilon > 0 \quad \text{and} \quad x \in B_0(2\rho),$$

$$\left|\frac{y_\epsilon}{|y_\epsilon|} + \frac{l_\epsilon}{|y_\epsilon|}x - \frac{x_\epsilon}{|y_\epsilon|}\right|^{\frac{n-2}{2}} w_\epsilon(x) \leq \left|\frac{y_\epsilon}{|y_\epsilon|} - \frac{x_\epsilon}{|y_\epsilon|}\right|^{\frac{n-2}{2}} \qquad \text{for } \epsilon > 0 \quad \text{and} \quad x \in B_0(2\rho)$$

Since $\lim\limits_{\epsilon\to 0} \frac{l_\epsilon}{|y_\epsilon|} = \lim\limits_{\epsilon\to 0} \left(\frac{\lambda_\epsilon}{|y_\epsilon|}\right)^{\frac{2-s_\epsilon}{2}} = 0$ from (4.39), and since $\lim\limits_{\epsilon\to 0} \frac{|x_\epsilon|}{|y_\epsilon|} = 0$, therefore we obtain that there exist a constant C_0 such that for $\epsilon > 0$ small

$$0 \leq w_\epsilon(x) \leq C_0 \qquad \text{for } x \in B_0(2\rho)$$

Since $w_\epsilon \in L^\infty\left(B_0(2\rho)\right)$, by *standard elliptic estimates* (see for instance [**14**]) from (4.46) it follows that there exists $w_0 \in C^1\left(B_0(2\rho)\right)$ such that up to a subsequence

$$\lim_{\epsilon\to 0} w_\epsilon = w_0 \qquad \text{in } C^1\left(B_0(\rho)\right)$$

And in particular we have $w_0(0) = 1$.

We have for $\epsilon > 0$ with a change of variable

$$\int_{B_{y_\epsilon}(l_\epsilon\rho)} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\, dx = \left(\frac{|y_\epsilon|^{s_\epsilon}}{\lambda_\epsilon^{s_\epsilon}}\right)^{\frac{n-2}{2}} \int_{B_0(\rho)} \frac{|w_\epsilon(x)|^{2^*(s_\epsilon)}}{\left|\frac{y_\epsilon}{|y_\epsilon|} + \frac{l_\epsilon}{|y_\epsilon|}x\right|^{s_\epsilon}}\, dx$$

Passing to the limit as $\epsilon \to 0$, we have

$$\int_{B_0(\rho)} w_0^{2^*}\, dx \leq \lim_{\epsilon\to 0}\left(\frac{\lambda_\epsilon^{s_\epsilon}}{|y_\epsilon|^{s_\epsilon}}\right)^{\frac{n-2}{2}} \limsup_{\epsilon\to 0}\int_{\Omega} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\, dx = 0$$

A contradiction since $w_0(0) = 1$. This completes Case 2.1.

Case 2.2: Suppose that

$$\lim_{\epsilon\to 0}\frac{d(y_\epsilon, \partial\Omega)}{l_\epsilon} = 0$$

Then

$$\lim_{\epsilon\to 0} y_\epsilon = y_0 \in \partial\Omega$$

Let $\mathcal{T}$ be a parametrisation of the boundary $\partial\Omega$ as in (4.18) around the point $p = y_0$. For all $\epsilon > 0$ let

$$\tilde{u}_\epsilon = u_\epsilon \circ \mathcal{T} \qquad \text{on } U \cap \{x_1 \leq 0\}$$

For $i, j = 1, \ldots, n$, we let $g_{ij} = (\partial_i\mathcal{T}, \partial_j\mathcal{T})$ be the metric induced by the chart $\mathcal{T}$ on the domain $U \cap \{x_1 < 0\}$, and let Δ_g denote the Laplace-Beltrami operator with respect to the metric g. From equation (4.4) it follows that for any $\epsilon > 0$, $\tilde{u}_\epsilon$ satisfies weakly the equation

$$\begin{cases} \Delta_g \tilde{u}_\epsilon + a\circ\mathcal{T}(x)\tilde{u}_\epsilon = \frac{\tilde{u}_\epsilon^{2^*(s_\epsilon)-1}}{|\mathcal{T}(x)|^{s_\epsilon}} & \text{in } U \cap \{x_1 < 0\} \\ \tilde{u}_\epsilon = 0 & \text{on } U \cap \{x_1 = 0\} \end{cases}$$

Let $z'_\epsilon \in \partial\Omega$ be such that

$$|z'_\epsilon - y_\epsilon| = d(y_\epsilon, \partial\Omega) \qquad \text{for } \epsilon > 0$$

And let $\tilde{y}_\epsilon$, $\tilde{z}'_\epsilon \in U$ be such that

$$\mathcal{T}(\tilde{y}_\epsilon) = y_\epsilon \qquad \text{and} \qquad \mathcal{T}(\tilde{z}'_\epsilon) = z'_\epsilon$$

Then it follows from the properties of the boundary chart $\mathcal{T}$, that

$$\lim_{\epsilon\to 0}\tilde{y}_\epsilon = 0 = \lim_{\epsilon\to 0}\tilde{z}'_\epsilon\ , \qquad (\tilde{y}_\epsilon)_1 < 0 \ \text{ and } (\tilde{z}'_\epsilon)_1 = 0$$

For $\epsilon > 0$ we set

$$\tilde{w}_\epsilon = \frac{\tilde{u}_\epsilon\left(\tilde{z}'_\epsilon + l_\epsilon x\right)}{\tilde{u}_\epsilon(\tilde{y}_\epsilon)} \qquad \text{for } x \in \frac{U - \tilde{z}'_\epsilon}{l_\epsilon} \cap \{x_1 \leq 0\}$$

So for any $R > 0$, $\tilde{w}_\epsilon$ is defined on $B_0(R) \cap \{x_1 \leq 0\}$ for $\epsilon > 0$ small enough. Let

$$\tilde{g}_\epsilon = g\left(\tilde{z}'_\epsilon + l_\epsilon x\right)$$

Then from equation (4.4) it follows that for $\epsilon > 0$ small, $\tilde{w}_\epsilon$ satisfies weakly the equation

$$\begin{cases} \Delta_{\tilde{g}_\epsilon}\tilde{w}_\epsilon + l_\epsilon^2\left(a\circ\mathcal{T}\left(\tilde{z}'_\epsilon + l_\epsilon x\right)\right)\tilde{w}_\epsilon = \frac{\tilde{w}_\epsilon^{2^*(s_\epsilon)-1}}{\left|\frac{\mathcal{T}\left(\tilde{z}'_\epsilon + l_\epsilon x\right)}{|y_\epsilon|}\right|^{s_\epsilon}} & \text{in } B_0(R) \cap \{x_1 < 0\} \\ \tilde{w}_\epsilon = 0 & \text{on } B_0(R) \cap \{x_1 = 0\} \end{cases} \tag{4.47}$$

From the properties of the boundary chart $\mathcal{T}$ it follows that for $\epsilon > 0$ small

$$\mathcal{T}\left(\tilde{z}'_\epsilon + l_\epsilon x\right) = y_\epsilon + O_R(1)l_\epsilon \qquad \text{for } x \in B_0(R) \cap \{x_1 \leq 0\}$$

where

$$|O_R(1)| \leq C_R$$

for some $C_R > 0$. Then $\lim_{\epsilon\to 0} \frac{l_\epsilon}{|y_\epsilon|} = \lim_{\epsilon\to 0} \left(\frac{\lambda_\epsilon}{|y_\epsilon|}\right)^{\frac{2-s_\epsilon}{2}} = 0$ from (4.39). Therefore

$$\lim_{\epsilon\to 0} \left|\frac{\mathcal{T}\left(\tilde{z}'_\epsilon + l_\epsilon x\right)}{|y_\epsilon|}\right|^{s_\epsilon} = 1 \qquad \text{in } C^0\left(B_0(R)\cap\{x_1\leq 0\}\right)$$

And then eqn (4.47) then can be written as

$$\begin{cases} \Delta_{\tilde{g}_\epsilon} \tilde{w}_\epsilon + l_\epsilon^2 \left(a\circ\mathcal{T}\left(\tilde{z}'_\epsilon + l_\epsilon x\right)\right)\tilde{w}_\epsilon = (1+o(1))\,\tilde{w}_\epsilon^{2^*(s_\epsilon)-1} & \text{in } B_0(R)\cap\{x_1<0\} \\ \quad (4.48) & \\ \qquad\qquad \text{with} \qquad \tilde{w}_\epsilon = 0 & \text{on } B_0(R)\cap\{x_1=0\} \end{cases}$$

where $\lim_{\epsilon\to 0} o(1) = 0$ in $C^0\left(B_0(R)\cap\{x_1\leq 0\}\right)$. We have

$$|\mathcal{T}\left(\tilde{z}'_\epsilon + l_\epsilon x\right) - x_\epsilon|^{\frac{n-2}{2}}\,\tilde{w}_\epsilon(x) \leq |y_\epsilon - x_\epsilon|^{\frac{n-2}{2}}\,\lambda_\epsilon^{\frac{n-2}{2}}\,u_\epsilon(y_\epsilon) \qquad \text{for } \epsilon>0 \quad \text{and} \quad x\in B_0(R)\cap\{x_1<0\},$$

$$\left|\frac{\mathcal{T}\left(\tilde{z}'_\epsilon + l_\epsilon x\right)}{|y_\epsilon|} - \frac{x_\epsilon}{|y_\epsilon|}\right|^{\frac{n-2}{2}} \tilde{w}_\epsilon(x) \leq \left|\frac{y_\epsilon}{|y_\epsilon|} - \frac{x_\epsilon}{|y_\epsilon|}\right|^{\frac{n-2}{2}} \qquad \text{for } \epsilon>0 \quad \text{and} \quad x\in B_0(R)\cap\{x_1<0\}$$

Since $\lim_{\epsilon\to 0} \frac{l_\epsilon}{|y_\epsilon|} = \lim_{\epsilon\to 0} \left(\frac{\lambda_\epsilon}{|y_\epsilon|}\right)^{\frac{2-s_\epsilon}{2}} = 0$ and since $\lim_{\epsilon\to 0} \frac{|x_\epsilon|}{|y_\epsilon|} = 0$, therefore $\lim_{\epsilon\to 0} \left|\frac{\mathcal{T}\left(\tilde{z}'_\epsilon + l_\epsilon x\right)}{|y_\epsilon|} - \frac{x_\epsilon}{|y_\epsilon|}\right| = 1$ in $C^0\left(B_0(R)\cap\{x_1\leq 0\}\right)$ and therefore, there exist a constant C_0 such that for $\epsilon > 0$ small

$$0 \leq \tilde{w}_\epsilon(x) \leq C_0 \qquad \text{for } x\in B_0(2\rho)$$

By *standard elliptic estimates* (see for instance [**14**]) it follows that there exists $\tilde{w}_0 \in C^1\left(B_0(B_0(R)\cap\{x_1\leq 0\}\right)$ such that up to a subsequence

$$\lim_{\epsilon\to 0} \tilde{w}_\epsilon = \tilde{w}_0 \qquad \text{in } C^1\left(B_0(R/2)\cap\{x_1\leq 0\}\right)$$

And therefore in particular

$$\tilde{w}_0 \equiv 0 \qquad \text{on } B_0(R/2)\ \cap\{x_1=0\}$$

We have that

$$\tilde{w}_\epsilon\left(\frac{\tilde{y}_\epsilon - \tilde{z}'_\epsilon}{l_\epsilon}\right) = 1$$

From the properties of the boundary chart $\mathcal{T}$ it follows that, for all $\epsilon > 0$

$$\frac{|\tilde{y}_\epsilon - \tilde{z}'_\epsilon|}{l_\epsilon} = O\left(\frac{|y_\epsilon - z'_\epsilon|}{l_\epsilon}\right)$$

So if $\frac{d(y_\epsilon,\partial\Omega)}{l_\epsilon} = 0$ as $\epsilon\to 0$, then $\frac{\tilde{y}_\epsilon - \tilde{z}'_\epsilon}{l_\epsilon} \longrightarrow 0$ as $\epsilon\to 0$. And we have

$$\tilde{w}_0(0) = 1$$

A contradiction. This ends Case 2.2 and then Step 2 by proving (4.45). $\square$

Since $\lim_{\epsilon\to 0} \frac{|y_\epsilon - x_\epsilon|}{\lambda_\epsilon} = +\infty$, we then also have with (4.45) that

$$\lim_{\epsilon\to 0} \frac{|y_\epsilon - x_\epsilon|}{l_\epsilon} = \lim_{\epsilon\to 0} \frac{|y_\epsilon - x_\epsilon|}{\lambda_\epsilon} \frac{\lambda_\epsilon^{s_\epsilon/2}}{|y_\epsilon|^{s_\epsilon/2}} = +\infty \tag{4.49}$$

Step 3: Suppose that

$$\frac{d(y_\epsilon, \partial\Omega)}{l_\epsilon} = O(1) \qquad \text{as } \epsilon \to 0 \tag{4.50}$$

Then

$$\lim_{\epsilon\to 0} y_\epsilon = y_0 \in \partial\Omega$$

Step 3.1: Let $\mathcal{T}$ be a parametrisation of the boundary $\partial\Omega$ as in (4.18) around the point $p = y_0$. For $\epsilon > 0$ let

$$\tilde{u}_\epsilon = u_\epsilon \circ \mathcal{T} \qquad \text{on } U \cap \{x_1 \leq 0\}$$

For $i, j = 1, \dots, n$, we let $g_{ij} = (\partial_i \mathcal{T}, \partial_j \mathcal{T})$ be the metric induced by the chart $\mathcal{T}$ on the domain $U \cap \{x_1 < 0\}$, and let Δ_g denote the Laplace-Beltrami operator with respect to the metric g. From equation (4.4) it follows that for any $\epsilon > 0$, $\tilde{u}_\epsilon$ satisfies weakly the equation

$$\begin{cases} \Delta_g \tilde{u}_\epsilon + a \circ \mathcal{T}(x) \tilde{u}_\epsilon = \frac{\tilde{u}_\epsilon^{2^*(s_\epsilon)-1}}{|\mathcal{T}(x)|^{s_\epsilon}} & \text{in } U \cap \{x_1 < 0\} \\ \tilde{u}_\epsilon = 0 & \text{on } U \cap \{x_1 = 0\} \end{cases}$$

Let $z'_\epsilon \in \partial\Omega$ be such that

$$\left|z'_{,\epsilon} - y_\epsilon\right| = d(y_\epsilon, \partial\Omega) \qquad \text{for } \epsilon > 0$$

And let $\tilde{y}_\epsilon$, $\tilde{z}'_\epsilon \in U$ be such that

$$\mathcal{T}(\tilde{y}_\epsilon) = y_\epsilon \qquad \text{and} \qquad \mathcal{T}(\tilde{z}'_\epsilon) = z'_\epsilon$$

Then it follows from the properties of the boundary chart $\mathcal{T}$, that

$$\lim_{\epsilon\to 0} \tilde{y}_\epsilon = 0 = \lim_{\epsilon\to 0} \tilde{z}'_\epsilon\ , \qquad (\tilde{y}_\epsilon)_1 < 0 \ \text{ and } (\tilde{z}'_\epsilon)_1 = 0$$

For $\epsilon > 0$ we set

$$\tilde{w}_\epsilon(x) = \frac{\tilde{u}_\epsilon\left(\tilde{z}'_\epsilon + l_\epsilon x\right)}{\tilde{u}_\epsilon(\tilde{y}_\epsilon)} \qquad \text{for } x \in \frac{U - \tilde{z}'_\epsilon}{l_\epsilon} \cap \{x_1 \leq 0\}$$

Step 3.2: For any $\eta \in C_c^\infty(\mathbb{R}^n)$, one has that $\eta\tilde{w}_\epsilon \in H_{1,0}^2(\mathbb{R}^n_-)$ for $\epsilon > 0$ sufficiently small. Let $x \in \mathbb{R}^n_-$, then

$$\nabla\left(\eta\tilde{w}_\epsilon\right)(x) = \tilde{w}_\epsilon(x)\nabla\eta(x) + \frac{l_\epsilon}{u_\epsilon(y_\epsilon)}\ \eta(x) D_{(\tilde{z}'_\epsilon + l_\epsilon x)}\mathcal{T}\left[\nabla u_\epsilon\left(\mathcal{T}(\tilde{z}'_\epsilon + l_\epsilon x)\right)\right]$$

For any $\theta > 0$, there exists $C(\theta) > 0$ such that for any $a, b > 0$

$$(a+b)^2 \leq C(\theta)a^2 + (1+\theta)b^2$$

With this inequality we then obtain

$$\int_{\mathbb{R}^n_-} \left|\nabla\left(\eta\tilde{w}_\epsilon\right)\right|^2\ dx \leq C(\theta)\int_{\mathbb{R}^n_-} |\nabla\eta|^2 \tilde{w}_\epsilon^2\ dx + (1+\theta)\frac{l_\epsilon^2}{u_\epsilon^2(y_\epsilon)}\int_{\mathbb{R}^n_-} \eta^2\left|D_{(\tilde{z}'_\epsilon + l_\epsilon x)}\mathcal{T}\left[\nabla u_\epsilon\left(\mathcal{T}(\tilde{z}'_\epsilon + l_\epsilon x)\right)\right]\right|^2\ dx$$

Since $D_0\mathcal{T} = \mathbb{I}_{\mathbb{R}^n}$, we have for $\epsilon > 0$ sufficiently small

$$\int_{\mathbb{R}^n_-} |\nabla(\eta\tilde{w}_\epsilon)|^2 \; dx \leq C(\theta) \int_{\mathbb{R}^n_-} |\nabla\eta|^2 \tilde{w}_\epsilon^2 \; dx + (1+\theta)\left(1 + O(l_\epsilon) + O(\tilde{z}'_\epsilon)\right) \frac{l_\epsilon^2}{u_\epsilon^2(y_\epsilon)} \int_{\mathbb{R}^n_-} \eta^2 \left|\nabla u_\epsilon\left(\mathcal{T}(\tilde{z}'_\epsilon + l_\epsilon x)\right)\right|^2 \; dx$$

With Hölder inequality and a change of variables this becomes

$$\int_{\mathbb{R}^n_-} |\nabla(\eta\tilde{w}_\epsilon)|^2 \; dx \leq C(\theta) \left(\frac{\lambda_\epsilon}{l_\epsilon}\right)^{n-2} \|\nabla\eta\|^2_{L^n} \left(\int_\Omega u_\epsilon^{2^*} \; dx\right)^{\frac{n-2}{n}} + (1+\theta+O(l_\epsilon)+O(\tilde{z}'_\epsilon)) \left(\frac{\lambda_\epsilon}{l_\epsilon}\right)^{n-2} \int_{\mathbb{R}^n} |\nabla u_\epsilon|^2$$

Then by the *Sobolev inequality* (4.7) we obtain for $\epsilon > 0$ small enough

$$\int_{\mathbb{R}^n_-} |\nabla(\eta\tilde{w}_\epsilon)|^2 \; dx \leq \left[C(\theta)\|\nabla\eta\|^2_{L^n} + (1+\theta+O(l_\epsilon)+O(\tilde{z}'_\epsilon))\right] \left(\frac{\lambda_\epsilon}{|y_\epsilon|}\right)^{\frac{n-2}{2}s_\epsilon} \int_{\mathbb{R}^n} |\nabla u_\epsilon|^2 \; dx$$

$$\leq \left[C(\theta)\|\nabla\eta\|^2_{L^n} + (1+\theta+O(l_\epsilon)+O(\tilde{z}'_\epsilon))\right] \int_{\mathbb{R}^n} |\nabla u_\epsilon|^2 \; dx \tag{4.51}$$

$$\text{since from (4.39) } \lim_{\epsilon\to 0} \frac{|y_\epsilon|}{\lambda_\epsilon} = +\infty$$

Now $\|u_\epsilon\|_{H^2_{1,0}(\Omega)} = O(1)$ and $l_\epsilon \to 0$ as $\epsilon \to 0$, so for $\epsilon > 0$ small enough

$$\|\eta\tilde{w}_\epsilon\|_{\mathscr{D}^{1,2}(\mathbb{R}^n_-)} \leq C_\eta$$

Where C_η is a constant depending on the function η. It then follows that there exists $w_\eta \in \mathscr{D}^{1,2}(\mathbb{R}^n_-)$ such that upto a subsequence

$$\begin{cases} \eta\tilde{w}_\epsilon \rightharpoonup \tilde{w}_\eta & \text{weakly in } \mathscr{D}^{1,2}(\mathbb{R}^n_-) \text{ as } \epsilon \to 0 \\ \eta\tilde{w}_\epsilon(x) \to \tilde{w}_\eta(x) & a.e\ x \text{ in } \mathbb{R}^n_- \text{ as } \epsilon \to 0 \end{cases}$$

Step 3.3: Let $\eta_1 \in C_c^\infty(\mathbb{R}^n)$, $0 \leq \eta_1 \leq 1$ be a smooth cut-off function, such that

$$\eta_1 = \begin{cases} 1 & \text{for } x \in B_0(1) \\ 0 & \text{for } x \in \mathbb{R}^n \setminus B_0(2) \end{cases}$$

For any $R > 0$ we let $\eta_R = \eta_1(x/R)$. Then with a diagonal argument we can assume that, upto a subsequence for any $\epsilon > 0$, there exists $\tilde{w}_R \in \mathscr{D}^{1,2}(\mathbb{R}^n_-)$ such that

$$\begin{cases} \eta_R\tilde{w}_\epsilon \rightharpoonup \tilde{w}_R & \text{weakly in } \mathscr{D}^{1,2}(\mathbb{R}^n_-) \text{ as } \epsilon \to 0 \\ \eta_R\tilde{w}_\epsilon \to \tilde{w}_R & a.e \text{ in } \mathbb{R}^n_- \text{ as } \epsilon \to 0 \end{cases}$$

Since $\|\nabla\eta_R\|^2_n = \|\nabla\eta_1\|^2_n$ for all $R > 0$, letting $\epsilon \to 0$ in (4.51) we obtain that

$$\int_{\mathbb{R}^n_-} |\nabla\tilde{w}_R|^2 \, dx \leq C \qquad \text{for all } R > 0$$

where C is a constant independent of R. So there exists $\tilde{w} \in \mathscr{D}^{1,2}(\mathbb{R}^n_-)$ such that

$$\begin{cases} \tilde{w}_R \rightharpoonup \tilde{w} & \text{weakly in } \mathscr{D}^{1,2}(\mathbb{R}^n_-) \text{ as } R \to \infty \\ \tilde{w}_R(x) \to \tilde{w}(x) & a.e\ x \text{ in } \mathbb{R}^n_- \text{ as } R \to \infty \end{cases}$$

Step 3.4: We claim that $\tilde{w} \in C^1(\overline{\mathbb{R}^n_-})$ and it satisfies weakly the equation

$$\begin{cases} \Delta\tilde{w} = \tilde{w}^{2^*-1} & \text{in } \mathbb{R}^n_- \\ \tilde{w} = 0 & \text{on } \{x_1 = 0\} \end{cases}$$

Let

$$\tilde{g}_\epsilon = g\left(\tilde{z}'_\epsilon + l_\epsilon x\right)$$

Then from eqn (4.4) it follows that for any $\epsilon > 0$ and $R > 0$, $\eta_R \tilde{w}_\epsilon$ satisfies weakly the equation

$$\begin{cases} \Delta_{\tilde{g}_\epsilon}\left(\eta_R \tilde{w}_\epsilon\right) + l_\epsilon^2 \left(a \circ \mathcal{T}\left(\tilde{z}'_\epsilon + l_\epsilon x\right)\right)\left(\eta_R \tilde{w}_\epsilon\right) = \dfrac{\left(\eta_R \tilde{w}_\epsilon\right)^{2^*(s_\epsilon)-1}}{\left|\frac{\mathcal{T}\left(\tilde{z}'_\epsilon + l_\epsilon x\right)}{|y_\epsilon|}\right|^{s_\epsilon}} & \text{in } B_0(R) \cap \{x_1 < 0\} \\ \qquad (4.52) & \\ \eta_R \tilde{w}_\epsilon = 0 & \text{on } B_0(R) \cap \{x_1 = 0\} \end{cases}$$

From the properties of the boundary chart $\mathcal{T}$ it follows that for $\epsilon > 0$ small

$$\mathcal{T}\left(\tilde{z}'_\epsilon + l_\epsilon x\right) = y_\epsilon + O_R(1) l_\epsilon \qquad \text{for } x \in B_0(R) \cap \{x_1 \leq 0\}$$

where

$$|O_R(1)| \leq C_R$$

for some $C_R > 0$. Then $\lim\limits_{\epsilon \to 0} \dfrac{l_\epsilon}{|y_\epsilon|} = \lim\limits_{\epsilon \to 0} \left(\dfrac{\lambda_\epsilon}{|y_\epsilon|}\right)^{\frac{2-s_\epsilon}{2}} = 0$ since $\lim\limits_{\epsilon \to 0} \dfrac{|y_\epsilon|}{\lambda_\epsilon} = +\infty$, as we have shown earlier. Therefore

$$\lim_{\epsilon \to 0} \left|\frac{\mathcal{T}\left(\tilde{z}'_\epsilon + l_\epsilon x\right)}{|y_\epsilon|}\right|^{s_\epsilon} = 1 \qquad \text{in } C^0\left(B_0(R) \cap \{x_1 \leq 0\}\right)$$

And then equation (4.52) then can be written as

$$\begin{cases} \Delta\left(\eta_R \tilde{w}_\epsilon\right) + l_\epsilon^2 \left(a \circ \mathcal{T}\left(\tilde{z}'_\epsilon + l_\epsilon x\right)\right)\left(\eta_R \tilde{w}_\epsilon\right) = (1 + o(1))\left(\eta_R \tilde{w}_\epsilon\right)^{2^*(s_\epsilon)-1} & \text{in } B_0(R) \cap \{x_1 < 0\} \\ \qquad (4.53) & \\ \text{with} \qquad \eta_R \tilde{w}_\epsilon = 0 & \text{on } B_0(R) \cap \{x_1 = 0\} \end{cases}$$

where $\lim\limits_{\epsilon \to 0} o(1) = 0$ in $C^0\left(B_0(R) \cap \{x_1 \leq 0\}\right)$. For $R > 0$ and $\epsilon > 0$ we have

$$\left|\mathcal{T}\left(\tilde{z}'_\epsilon + l_\epsilon x\right) - x_\epsilon\right|^{\frac{n-2}{2}} \eta_R \tilde{w}_\epsilon(x) \leq |y_\epsilon - x_\epsilon|^{\frac{n-2}{2}} \lambda_\epsilon^{\frac{n-2}{2}} u_\epsilon(y_\epsilon),$$

$$\left(\frac{\left|\mathcal{T}\left(\tilde{z}'_\epsilon + l_\epsilon x\right) - x_\epsilon\right|}{|y_\epsilon - x_\epsilon|}\right)^{\frac{n-2}{2}} \eta_R \tilde{w}_\epsilon(x) \leq 1$$

Since $\lim\limits_{\epsilon \to 0} \frac{|y_\epsilon - x_\epsilon|}{l_\epsilon} = +\infty$, we obtain

$$\lim_{\epsilon \to 0} \frac{|\mathcal{T}(\tilde{z}'_\epsilon + l_\epsilon x) - x_\epsilon|}{|y_\epsilon - x_\epsilon|} = \lim_{\epsilon \to 0} \frac{|y_\epsilon - x_\epsilon + O_R(1) l_\epsilon|}{|y_\epsilon - x_\epsilon|} = 1 \qquad \text{for all } x \in B_0(R) \cap \{x_1 \leq 0\}$$

It then follows that for $\epsilon > 0$ sufficiently small

$$\eta_R \tilde{w}_\epsilon(x) \leq 2 \qquad \text{for all } x \in B_0(R) \cap \{x_1 \leq 0\}$$

By standard elliptic estimates (see for instance [**14**]) then it follows that the sequence $\left(\eta_R \tilde{w}_\epsilon\right)_{\epsilon > 0}$ is bounded in
$C^{1,\alpha_0}\left(B_0(R) \cap \{x_1 \leq 0\}\right)$ for some $\alpha_0 \in (0,1)$. So by *Arzela-Ascoli's theorem* one has that $\tilde{w}_R \in C^{1,\alpha}\left(B_0(R/2) \cap \{x_1 \leq 0\}\right)$ for $0 < \alpha < \alpha_0$, and that, up to a subsequence

$$\lim_{\epsilon \to 0} \eta_R \tilde{w}_\epsilon = \tilde{w}_R \qquad \text{in } C^{1,\alpha}\left(B_0(R/4) \cap \{x_1 \leq 0\}\right)$$

for $0 < \alpha < \alpha_0$. And therefore in particular

$$\tilde{w}_R \equiv 0 \qquad \text{on } B_0(R/4) \cap \{x_1 = 0\} \tag{4.54}$$

Letting $\epsilon \to 0$ in eqn (4.53) gives that $\tilde{w}_R$ satisfies weakly the equation

$$\begin{cases} \Delta \tilde{w}_R = \tilde{w}_R^{2^*-1} & \text{in } B_0(R/4) \cap \{x_1 \le 0\} \\ \tilde{w}_R = 0 & \text{on } B_0(R/4) \cap \{x_1 = 0\} \end{cases} \tag{4.55}$$

We have that: $0 \le \tilde{w}_R \le 2$, so again from standard elliptic estimates and applying the *Arzela-Ascoli's theorem* it follows that $\tilde{w} \in C^1(\overline{\mathbb{R}^n_-})$ and $\lim_{R\to+\infty} \tilde{w}_R = \tilde{w}$ in $C^1_{loc}(\overline{\mathbb{R}^n_-})$ up to a subsequence. Moreover letting $R \to +\infty$ we obtain that

$$\begin{cases} \Delta \tilde{w} = \tilde{w}^{2^*-1} & \text{in } \mathbb{R}^n_- \\ \tilde{w} \ge 0 & \text{in } \mathbb{R}^n_- \\ \tilde{w} = 0 & \text{on } \{x_1 = 0\} \end{cases}$$

This proves the claim and ends Step 3.4.

Step 3.5: We have that

$$\tilde{w}_\epsilon \left(\frac{\tilde{y}_\epsilon - \tilde{z}'_\epsilon}{l_\epsilon} \right) = 1$$

From the properties of the boundary chart $\mathcal{T}$ it follows that, for all $\epsilon > 0$

$$\frac{|\tilde{y}_\epsilon - \tilde{z}'_\epsilon|}{l_\epsilon} = O\left(\frac{|y_\epsilon - z'_\epsilon|}{l_\epsilon} \right)$$

So if $\dfrac{d\,(y_\epsilon, \partial\Omega)}{l_\epsilon} = O(1)$ as $\epsilon \to 0$, then there exists $\tilde{y} \in \overline{\mathbb{R}^n_-}$ such that

$$\frac{\tilde{y}_\epsilon - \tilde{z}'_\epsilon}{l_\epsilon} \longrightarrow \tilde{y} \qquad as\ \epsilon \to 0$$

For $R > 0$ sufficiently large we have

$$\tilde{w}_R(\tilde{y}) = \lim_{\epsilon\to 0} (\eta_R \tilde{w}_\epsilon) \left(\frac{\tilde{y}_\epsilon - \tilde{z}'_\epsilon}{l_\epsilon} \right) = 1$$

and therefore

$$\tilde{w}(\tilde{y}) = \lim_{R\to+\infty} \tilde{w}_R(\tilde{y}) = 1$$

From (4.54) it follows that $\tilde{y} \in \mathbb{R}^n_-$. But then this implies $\tilde{w} \in C^1(\overline{\mathbb{R}^n_-})$ is a nontrivial weak solution of the equation

$$\begin{cases} \Delta \tilde{w} = \tilde{w}^{2^*-1} & \text{in } \mathbb{R}^n_- \\ \tilde{w} = 0 & \text{on } \{x_1 = 0\} \end{cases}$$

which is a contradiction, see Struwe's book [**18**] (Chapter III , theorem 1.3). This proves proposition 4.5.1 when (4.50) holds. This ends Step 3.

Step 4: Suppose that

$$\lim_{\epsilon\to 0} \frac{d\,(y_\epsilon, \partial\Omega)}{l_\epsilon} = +\infty \tag{4.56}$$

For $\epsilon > 0$ we let

$$w_\epsilon(x) = \frac{u_\epsilon(y_\epsilon + l_\epsilon x)}{u_\epsilon(y_\epsilon)} \qquad \text{for } x \in \frac{\Omega - y_\epsilon}{l_\epsilon}$$

Step 4.1: For any $\eta \in C_c^\infty(\mathbb{R}^n)$, one has that $\eta w_\epsilon \in H_1^2(\mathbb{R}^n)$ for $\epsilon > 0$ sufficiently small. We claim that for any $\eta \in C_c^\infty(\mathbb{R}^n)$, there exists $w_\eta \in \mathscr{D}^{1,2}(\mathbb{R}^n)$ such that upto a subsequence

$$\eta w_\epsilon \rightharpoonup w_\eta \qquad \text{weakly in } \mathscr{D}^{1,2}(\mathbb{R}^n) \qquad \text{as } \epsilon \to 0$$

Let $x \in \mathbb{R}^n$, then for $\epsilon > 0$

$$\nabla\left(\eta w_\epsilon\right)(x) = w_\epsilon \nabla\eta(x) + \lambda_\epsilon^{\frac{n-2}{2}} l_\epsilon \ \eta \ \nabla u_\epsilon(y_\epsilon + l_\epsilon x)$$

For any $\theta > 0$, there exists $C(\theta) > 0$ such that for any $x, y > 0$

$$(x+y)^2 \le C(\theta)x^2 + (1+\theta)y^2$$

With the help of the above inequality we then obtain

$$\int_{\mathbb{R}^n} |\nabla\left(\eta w_\epsilon\right)|^2 \ dx \le C(\theta) \int_{\mathbb{R}^n} |\nabla\eta|^2 w_\epsilon^2 \ dx + (1+\theta)\lambda_\epsilon^{n-2} l_\epsilon^2 \int_{\mathbb{R}^n} \eta^2 \left|\nabla u_\epsilon(y_\epsilon + l_\epsilon x)\right|^2 \ dx$$

With Hölder inequality and a change of variables this becomes

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla\left(\eta w_\epsilon\right)|^2 \ dx \le & \left(\frac{\lambda_\epsilon}{l_\epsilon}\right)^{n-2} C(\theta) \left\|\nabla\eta\right\|_{L^n}^2 \left(\int_\Omega u_\epsilon^{2^*} \ dx\right)^{\frac{n-2}{n}} \\ & + (1+\theta)\left(\frac{\lambda_\epsilon}{l_\epsilon}\right)^{n-2} \int_\Omega \left(\eta\left(\frac{x-y_\epsilon}{l_\epsilon}\right)\right)^2 |\nabla u_\epsilon|^2 \ dx \end{aligned} \tag{4.57}$$

By *Sobolev inequality* (4.7) we obtain for $\epsilon > 0$ small enough

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla\left(\eta w_\epsilon\right)|^2 \ dx &\le \left[C(\theta)\left\|\nabla\eta\right\|_{L^n}^2 + (1+\theta)\sup\eta^2\right]\left(\frac{\lambda_\epsilon}{|y_\epsilon|}\right)^{\frac{n-2}{2}s_\epsilon} \int_{\mathbb{R}^n} |\nabla u_\epsilon|^2 \ dx \\ &\le \left[C(\theta)\left\|\nabla\eta\right\|_{L^n}^2 + (1+\theta)\sup\eta^2\right] \int_{\mathbb{R}^n} |\nabla u_\epsilon|^2 \ dx \\ &\text{since } \lim_{\epsilon\to 0}\frac{|y_\epsilon|}{\lambda_\epsilon} = +\infty \end{aligned}$$

Now $\|u_\epsilon\|_{H_{1,0}^2(\Omega)} = O(1)$ and $l_\epsilon \to 0$ as $\epsilon \to 0$, so for $\epsilon > 0$ small enough

$$\|\eta w_\epsilon\|_{\mathscr{D}^{1,2}(\mathbb{R}^n)} \le C_\eta$$

Where C_η is a constant depending on the function η. It then follows that there exists $v_\eta \in \mathscr{D}^{1,2}(\mathbb{R}^n)$ such that upto a subsequence

$$\begin{cases} \eta w_\epsilon \rightharpoonup w_\eta & \text{weakly in } \mathscr{D}^{1,2}(\mathbb{R}^n) \ \text{ as } \epsilon \to 0 \\ \eta w_\epsilon(x) \to w_\eta(x) & a.e \ x \ \text{ in } \mathbb{R}^n \ \text{ as } \epsilon \to 0 \end{cases} \tag{4.58}$$

Step 4.2: We claim that there exists $w \in \mathscr{D}^{1,2}(\mathbb{R}^n)$ such that for any $\eta \in C_c^\infty(\mathbb{R}^n)$ we have

$$w_\eta = \eta w$$

Let $\eta_1 \in C_c^\infty(\mathbb{R}^n)$, $0 \le \eta_1 \le 1$ be a smooth cut-off function, such that

$$\eta_1 = \begin{cases} 1 & \text{for} \quad x \in B_0(1) \\ 0 & \text{for} \quad x \in \mathbb{R}^n \backslash B_0(2) \end{cases}$$

For any $R > 0$ we let $\eta_R = \eta_1(x/R)$. Then with a diagonal argument we can assume that, upto a subsequence for any $\epsilon > 0$, there exists $\tilde{w}_R \in \mathscr{D}^{1,2}(\mathbb{R}^n)$ such that

$$\begin{cases} \eta_R w_\epsilon \rightharpoonup w_R & \text{weakly in } \mathscr{D}^{1,2}(\mathbb{R}^n) \quad \text{as } \epsilon \to 0 \\ \eta_R w_\epsilon \to w_R & a.e \quad \text{in } \mathbb{R}^n \quad \text{as } \epsilon \to 0 \end{cases}$$

Since $\|\nabla \eta_R\|_n^2 = \|\nabla \eta_1\|_n^2$ for all $R > 0$, letting $\epsilon \to 0$ in (4.57) we obtain that

$$\int_{\mathbb{R}^n} |\nabla w_R|^2 \, dx \le C \qquad \text{for all } R > 0$$

where C is a constant independent of R. So there exists $w \in \mathscr{D}^{1,2}(\mathbb{R}^n)$ such that

$$\begin{cases} w_R \rightharpoonup w & \text{weakly in } \mathscr{D}^{1,2}(\mathbb{R}^n) \quad \text{as } R \to \infty \\ w_R(x) \to w(x) & a.e \ x \quad \text{in } \mathbb{R}^n \quad \text{as } R \to \infty \end{cases}$$

And therefore for any $\eta \in C_c^\infty(\mathbb{R}^n)$

$$w_\eta = \eta w$$

This proves the claim.

Step 4.3: We claim that $w \in C^1(\mathbb{R}^n)$, $w \neq 0$ and it satisfies weakly the equation

$$\Delta w = w^{2^*-1} \qquad \text{in } \mathbb{R}^n$$

Using eqn (4.4) it follows that for any $\epsilon > 0$ and $R > 0$, $\eta_R w_\epsilon$ satisfies the equation

$$\Delta\left(\eta_R w_\epsilon\right) + l_\epsilon^2 a\left(y_\epsilon + l_\epsilon x\right)\left(\eta_R w_\epsilon\right) = \frac{\left(\eta_R w_\epsilon\right)^{2^*(s_\epsilon)-1}}{\left|\frac{y_\epsilon}{|y_\epsilon|} + \frac{l_\epsilon}{|y_\epsilon|}x\right|^{s_\epsilon}} \qquad \text{in } \mathscr{D}'(B_0(R)) \tag{4.59}$$

We have $\lim\limits_{\epsilon\to 0} \dfrac{l_\epsilon}{|y_\epsilon|} = \lim\limits_{\epsilon\to 0} \left(\dfrac{\lambda_\epsilon}{|y_\epsilon|}\right)^{\frac{2-s_\epsilon}{2}} = 0$. So we have

$$\lim_{\epsilon\to 0} \left|\frac{y_\epsilon}{|y_\epsilon|} + \frac{l_\epsilon}{|y_\epsilon|}x\right|^{s_\epsilon} = 1 \qquad \text{in } C^0\left(B_0(R)\right)$$

Then equation (4.59) then can be written as

(4.60)

$$\Delta\left(\eta_R w_\epsilon\right) + l_\epsilon^2 a\left(y_\epsilon + l_\epsilon x\right)\left(\eta_R w_\epsilon\right) = \left(1 + o(1)\right)\left(\eta_R w_\epsilon\right)^{2^*(s_\epsilon)-1} \qquad \text{in } \mathscr{D}'(B_0(R))$$

where $\lim\limits_{\epsilon\to 0} o(1) = 0$ in $C^0\left(B_0(R)\right)$. We have for $R > 0$ and $\epsilon > 0$

$$|y_\epsilon + l_\epsilon x - x_\epsilon|^{\frac{n-2}{2}} \ \eta_R w_\epsilon(x) \le |y_\epsilon - x_\epsilon|^{\frac{n-2}{2}} \lambda_\epsilon^{\frac{n-2}{2}} u_\epsilon(y_\epsilon),$$

$$\left(\frac{|y_\epsilon + l_\epsilon x - x_\epsilon|}{|y_\epsilon - x_\epsilon|}\right)^{\frac{n-2}{2}} \eta_R w_\epsilon(x) \le 1$$

Since $\lim\limits_{\epsilon\to 0} \frac{|y_\epsilon - x_\epsilon|}{l_\epsilon} = +\infty$, we obtain

$$\lim_{\epsilon\to 0} \frac{|y_\epsilon + l_\epsilon x - x_\epsilon|}{|y_\epsilon - x_\epsilon|} = 1 \qquad \text{for all } x \in B_0(R) \cap \{x_1 \le 0\}$$

It then follows that for $\epsilon > 0$ sufficiently small

$$\eta_R w_\epsilon(x) \le 2 \qquad \text{for all } x \in B_0(R) \cap \{x_1 \le 0\} \ \text{ uniformly}$$

It then follows from standard elliptic estimates (see for instance [**14**]) that $w_R \in C^1(B_0(R))$, and up to a subsequence

$$\lim_{\epsilon\to 0} \eta_R w_\epsilon = w_R \qquad \text{in } C^1_{loc}(B_0(R))$$

Letting $\epsilon \to 0$ in eqn (4.60) gives that w_R satisfies the equation

$$\Delta w_R = w_R^{2^*-1} \qquad \text{in } \mathscr{D}'(B_0(R))$$

Further as for any $\epsilon > 0$ and $R > 0$, $\eta_R w_\epsilon(0) = 1$, therefore $w_R(0) = 1$ for all $R > 0$. Again we have that: $0 \le w_R \le 2$ since $\eta_R w_\epsilon \to w_R$ $a.e$ in $\mathbb{R}^n$ as $\epsilon \to 0$. Then again from standard elliptic estimates it follows that $w \in C^1(\mathbb{R}^n)$ and $\lim_{R\to+\infty} w_R = w$ in $C^1_{loc}(\mathbb{R}^n)$ up to a subsequence. Moreover letting $R \to +\infty$ we obtain that

$$\Delta w = w^{2^*-1} \qquad \text{in } \mathscr{D}'(\mathbb{R}^n)$$

Moreover $w(0) = 1$ since $w_R(0) = 1$ for all $R > 0$, and so $w \not\equiv 0$. This proves teh claim and ends Step 4.3.

Step 4.4: We obtain by a change of variable for $R > 0$ and $\epsilon > 0$

$$\int_{B_{y_\epsilon}(Rl_\epsilon)} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\,dx = \left(\frac{|y_\epsilon|^{s_\epsilon}}{\lambda_\epsilon^{s_\epsilon}}\right)^{\frac{n-2}{2}} \int_{B_0(R)} \frac{|w_\epsilon(x)|^{2^*(s_\epsilon)}}{\left|\frac{y_\epsilon}{|y_\epsilon|} + \frac{l_\epsilon}{|y_\epsilon|}x\right|^{s_\epsilon}}\,dx$$

So

$$\int_{B_0(R)} \frac{|w_\epsilon(x)|^{2^*(s_\epsilon)}}{\left|\frac{y_\epsilon}{|y_\epsilon|} + \frac{l_\epsilon}{|y_\epsilon|}x\right|^{s_\epsilon}}\,dx = \left(\frac{\lambda_\epsilon^{s_\epsilon}}{|y_\epsilon|^{s_\epsilon}}\right)^{\frac{n-2}{2}} \int_{B_{y_\epsilon}(Rl_\epsilon)} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\,dx$$

Passing to the limit as $\epsilon \to 0$, we have for $R > 0$

$$\int_{B_0(R)} w^{2^*}\,dx \le \limsup_{\epsilon\to 0} \int_{B_{y_\epsilon}(Rl_\epsilon)} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\,dx$$

and so

$$\int_{\mathbb{R}^n} w^{2^*}\,dx = \lim_{R\to+\infty} \int_{B_0(R)} w^{2^*}\,dx \le \lim_{R\to+\infty} \limsup_{\epsilon\to 0} \int_{B_{y_\epsilon}(Rl_\epsilon)} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\,dx$$

Now for any $R > 0$, $B_{x_\epsilon}(Rk_\epsilon) \cap B_{y_\epsilon}(Rl_\epsilon) = \emptyset$ for $\epsilon > 0$ sufficiently small. For if $x \in B_{x_\epsilon}(Rk_\epsilon) \cap B_{y_\epsilon}(Rl_\epsilon)$, using that $\mu_\epsilon \le \lambda_\epsilon$ and (4.17), we get

$$\begin{aligned}
\frac{|y_\epsilon - x_\epsilon|}{l_\epsilon} \le \frac{|y_\epsilon - x|}{l_\epsilon} + \frac{|x - x_\epsilon|}{l_\epsilon} \le R\left(1 + \frac{k_\epsilon}{l_\epsilon}\right) &\le R\left(1 + \frac{|x_\epsilon|^{s_\epsilon/2}}{\mu_\epsilon^{s_\epsilon/2}}\frac{\lambda_\epsilon^{s_\epsilon/2}}{|y_\epsilon|^{s_\epsilon/2}}\frac{\mu_\epsilon}{\lambda_\epsilon}\right) \\
&\le R\left(1 + \frac{|x_\epsilon|^{s_\epsilon/2}}{\mu_\epsilon^{s_\epsilon/2}}\frac{\lambda_\epsilon^{s_\epsilon/2}}{|y_\epsilon|^{s_\epsilon/2}}\right) = O(R)
\end{aligned}$$

This is a contradiction since we have $\lim_{\epsilon\to 0} \dfrac{|y_\epsilon - x_\epsilon|}{l_\epsilon} = +\infty$ as shown in (4.49). Then by proposition 4.4.1

$$\int_{\mathbb{R}^n} w^{2^*}\,dx \le \lim_{R\to+\infty} \limsup_{\epsilon\to 0} \int_{\Omega\setminus B_{y_\epsilon}(Rl_\epsilon)} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\,dx = 0$$

But this contradicts what we have obtained in Step 4.3. Hence (4.38)) does not hold when (4.56) holds. This ends Step 4.

This completes the proof of Proposition 4.5.1. □

Having obtained the strong bound in Proposition 4.5.1 we show that

Proposition 4.5.2. *With the same hypothesis as in theorem 4.4 we have that there exists a constant $C > 0$ such that for $\epsilon > 0$*

$$|x - x_\epsilon|^{n/2} |\nabla u_\epsilon(x)| \leq C \quad \textit{and} \quad |x - x_\epsilon|^{n/2} u_\epsilon(x) \leq C d(x, \partial\Omega) \qquad \textit{for all } x \in \Omega$$

Proof. We proceed by contradiction and assume that there exists a sequence of points $(y_\epsilon)_{\epsilon>0}$ in Ω such that

$$|y_\epsilon - x_\epsilon|^{n/2} |\nabla u_\epsilon(y_\epsilon)| + \frac{|y_\epsilon - x_\epsilon|^{n/2} u_\epsilon(y_\epsilon)}{d(y_\epsilon, \partial\Omega)} \longrightarrow +\infty \qquad \text{as } \epsilon \to 0 \tag{4.61}$$

We let

$$\lim_{\epsilon \to 0} x_\epsilon = x_0 \in \overline{\Omega} \ \text{ and } \lim_{\epsilon \to 0} y_\epsilon = y_0 \in \overline{\Omega}$$

Case 1: we assume that $x_0 \neq y_0$. We choose $\delta > 0$ such that $0 < 4\delta < |x_0 - y_0|$. Then one has that $\delta < |x - x_\epsilon|$ for all $x \in B_{y_0}(2\delta) \cap \Omega$ and proposition 4.5.1 then gives us that there exists a constant $C(\delta) > 0$ such that

$$0 \leq u_\epsilon(x) \leq C(\delta) \qquad \text{for all } x \in B_{y_0}(2\delta) \cap \Omega$$

Further for $\epsilon > 0$, u_ϵ solves the equation

$$\begin{cases} \Delta u_\epsilon + a u_\epsilon = \frac{u_\epsilon^{2^*(s_\epsilon)-1}}{|x|^{s_\epsilon}} & \text{in } B_{y_0}(2\delta) \cap \Omega, \\ u_\epsilon = 0 & \text{on } B_{y_0}(2\delta) \cap \partial\Omega. \end{cases} \tag{4.62}$$

The right hand side of the above equation is uniformly bounded in $L^p\left(B_{y_0}(2\delta) \cap \Omega\right)$ for some $p > n$, for all $\epsilon > 0$ sufficiently small. Then from standard elliptic estimates (see for instance [**14**]) it follows that the sequence $(u_\epsilon)_{\epsilon>0}$ is bounded in $C^1\left(B_{y_0}(\delta) \cap \overline{\Omega}\right)$. So there exists a constant $C > 0$ such that

$$|\nabla u_\epsilon(x)| \leq C \qquad \text{and} \qquad u_\epsilon(x) \leq C d(x, \partial\Omega) \qquad \text{for all } x \in B_{y_0}(\delta) \cap \overline{\Omega}$$

a contradiction to (4.61), proving the proposition in Case 1.

Case 2: we assume that $x_0 = y_0$. Let

$$\alpha_\epsilon = |y_\epsilon - x_\epsilon|$$

Then $\lim_{\epsilon \to 0} \alpha_\epsilon = 0$.

Case 2.1: We assume that upto a subsequence

$$d(x_\epsilon, \partial\Omega) \geq 2 |y_\epsilon - x_\epsilon|$$

For $\epsilon > 0$ we let

$$\tilde{u}_\epsilon(x) = \alpha_\epsilon^{\frac{n-2}{2}} u_\epsilon(x_\epsilon + \alpha_\epsilon x) \qquad \text{for } x \in B_0(3/2)$$

This is well defined since $B_{x_\epsilon}(2\alpha_\epsilon) \subset \Omega$. Using lemma proposition 4.5.1 one obtains that there exists a constant $C > 0$ such that

$$\left(\frac{|x_\epsilon + \alpha_\epsilon x - x_\epsilon|}{\alpha_\epsilon}\right)^{\frac{n-2}{2}} \tilde{u}_\epsilon(x) \leq C \qquad \text{for } x \in B_0(3/2),$$

$$|x|^{\frac{n-2}{2}} \tilde{u}_\epsilon(x) \leq C \qquad \text{for } x \in B_0(3/2)$$

And so there exists a constant $C > 0$ such that for $\epsilon > 0$

$$\tilde{u}_\epsilon(x) \leq C \qquad \text{for all } x \in B_0(3/2) \setminus \overline{B_0(1/4)}$$

Moreover from equation (4.4) it follows that for $\epsilon > 0$, $\tilde{u}_\epsilon$ satisfies the equation

$$\Delta \tilde{u}_\epsilon + \alpha_\epsilon^2 a\left(y_\epsilon + \alpha_\epsilon x\right) \tilde{u}_\epsilon = \frac{\tilde{u}_\epsilon^{2^*(s_\epsilon)-1}}{\left|\frac{x_\epsilon}{\alpha_\epsilon} + x\right|^{s_\epsilon}} \qquad \text{in } \mathscr{D}'\left(B_0(3/2) \setminus \overline{B_0(1/4)}\right)$$

Since $0 \leq \tilde{u}_\epsilon(x) \leq C$ for all $x \in B_0(3/2) \setminus \overline{B_0(1/4)}$, the right hand side of the above equation is uniformly bounded in $L^p\left(B_0(3/2) \setminus \overline{B_0(1/4)}\right)$ for some $p > n$, for all $\epsilon > 0$ sufficiently small (the bound even holds in L^∞ when $|x_\epsilon/\alpha_\epsilon \to \infty$ as $\epsilon \to 0$). Then from standard elliptic estimates (see for instance [**14**]) it follows that

$$\|\tilde{u}_\epsilon\|_{C^1\left(B_0(5/4)\setminus\overline{B_0(1/2)}\right)} = O(1) \qquad \text{as } \epsilon \to 0$$

The points $\frac{y_\epsilon - x_\epsilon}{|y_\epsilon - x_\epsilon|} \in B_0(5/4) \setminus \overline{B_0(1/2)}$ for all $\epsilon > 0$. Taking $x = \frac{y_\epsilon - x_\epsilon}{|y_\epsilon - x_\epsilon|}$ one then obtains as $\epsilon \to 0$

$$\left|\nabla \tilde{u}_\epsilon\left(\frac{y_\epsilon - x_\epsilon}{|y_\epsilon - x_\epsilon|}\right)\right| = O(1), \qquad \tilde{u}_\epsilon\left(\frac{y_\epsilon - x_\epsilon}{|y_\epsilon - x_\epsilon|}\right) = O(1)$$

comig back to the defination of $\tilde{u}_\epsilon$ this implies that as $\epsilon \to 0$

$$|y_\epsilon - x_\epsilon|^{n/2} |\nabla u_\epsilon(y_\epsilon)| = O(1),$$

$$\frac{|y_\epsilon - x_\epsilon|^{n/2} u_\epsilon(y_\epsilon)}{d(x_\epsilon, \partial\Omega)} \leq \frac{|y_\epsilon - x_\epsilon|^{n/2} u_\epsilon(y_\epsilon)}{2|y_\epsilon - x_\epsilon|} = O(1)$$

But this is a contradiction to (4.61). This ends Case 2.1.

Case 2.2: We assume that upto a subsequence

$$d(x_\epsilon, \partial\Omega) \leq 2|y_\epsilon - x_\epsilon|$$

Let $\mathcal{T}: U \to V$ be a parametrisation of the boundary $\partial\Omega$ as in (4.18) around the point $p = x_0$. Let $z_\epsilon \in \partial\Omega$ be such that

$$|z_\epsilon - x_\epsilon| = d(x_\epsilon, \partial\Omega) \qquad \text{for } \epsilon > 0$$

And let $\tilde{x}_\epsilon$, $\tilde{z}_\epsilon \in U$ be such that

$$\mathcal{T}(\tilde{x}_\epsilon) = x_\epsilon \qquad \text{and} \qquad \mathcal{T}(\tilde{z}_\epsilon) = z_\epsilon$$

Then it follows from the properties of the boundary chart $\mathcal{T}$, that

$$\lim_{\epsilon \to 0} \tilde{x}_\epsilon = 0 = \lim_{\epsilon \to 0} \tilde{z}_\epsilon \ , \qquad (\tilde{x}_\epsilon)_1 < 0 \ \text{ and } (\tilde{z}_\epsilon)_1 = 0$$

For all $\epsilon > 0$, we let

$$\tilde{u}_\epsilon(x) = \alpha_\epsilon^{\frac{n-2}{2}} u_\epsilon \circ \mathcal{T}(\tilde{z}_\epsilon + \alpha_\epsilon x) \qquad \text{for } x \in \frac{U - \tilde{z}_\epsilon}{\alpha_\epsilon} \cap \{x_1 \leq 0\}$$

For any $R > 0$, $\tilde{u}_\epsilon$ is defined in $B_0(R) \cap \{x_1 \le 0\}$ for $\epsilon > 0$ small enough. Using lemma proposition 4.5.1 one obtains that there exists a constant $C > 0$ such that

$$\left(\frac{|\mathcal{T}(\tilde{z}_\epsilon + \alpha_\epsilon x) - x_\epsilon|}{\alpha_\epsilon} \right)^{\frac{n-2}{2}} \tilde{u}_\epsilon(x) \le C \qquad \text{for } x \in B_0(R) \cap \{x_1 \le 0\}$$

We let

$$\rho_\epsilon = \frac{\tilde{x}_\epsilon - \tilde{z}_\epsilon}{\alpha_\epsilon}$$

From the properities of the boundary map $\mathcal{T}$ it follows that $\rho_\epsilon \in \mathbb{R}^n_-$ and that

$$\frac{|\tilde{x}_\epsilon - \tilde{z}_\epsilon|}{\alpha_\epsilon} = O\left(\frac{|x_\epsilon - z_\epsilon|}{\alpha_\epsilon} \right) = O(1) \qquad \text{as } \epsilon \to 0$$

So there exists $\rho_0 \in \overline{\mathbb{R}}_-$ such that

$$\rho_\epsilon \to \rho_0 \qquad \text{as } \epsilon \to 0$$

Also from the properties of the boundary chart $\mathcal{T}$ it follows that there exist a constant $C_{\mathcal{T}} > 0$ such that

$$|\rho_\epsilon - x| \le C_{\mathcal{T}} \frac{|\mathcal{T}(\tilde{z}_\epsilon + \alpha_\epsilon x) - x_\epsilon|}{\alpha_\epsilon}$$

Therefore for some constant $C > 0$

$$|\rho_\epsilon - x|^{\frac{n-2}{2}} \tilde{u}_\epsilon(x) \le C \qquad \text{for } x \in B_0(R) \cap \{x_1 \le 0\}$$

Hence for any $R, \delta > 0$ there exist a constant $C(R, \delta)$ such that for $\epsilon > 0$ small

$$\tilde{u}_\epsilon(x) \le C(R, \delta) \qquad \text{for all } x \in B_0(R) \setminus \overline{B_{\rho_0}(\delta)} \cap \{x_1 \le 0\}$$

For $i, j = 1, \dots, n$, we let $g_{ij}(x) = (\partial_i \mathcal{T}(\tilde{z}_\epsilon + \alpha_\epsilon x), \partial_j \mathcal{T}(\tilde{z}_\epsilon + \alpha_\epsilon x))$, the induced metric on the domain $B_0(R) \cap \{x_1 < 0\}$, and let Δ_g denote the Laplace-Beltrami operator with respect to the metric g. From equation (4.4) it follows that for any $R, \delta > 0$, $\tilde{u}_\epsilon$ satisfies weakly the equation

$$\begin{cases} \Delta_{\tilde{g}_\epsilon} \tilde{u}_\epsilon + \alpha_\epsilon^2 \left(a \circ \mathcal{T}(\tilde{z}_\epsilon + \alpha_\epsilon x) \right) \tilde{u}_\epsilon = \frac{\tilde{u}_\epsilon(x)^{2^*(s_\epsilon) - 1}}{\left| \frac{\mathcal{T}(\tilde{z}_\epsilon + \alpha_\epsilon x)}{\alpha_\epsilon} \right|^{s_\epsilon}} & \text{in } B_0(R) \setminus \overline{B_{\rho_0}(\delta)} \cap \{x_1 < 0\} \\ \qquad (4.63) & \\ \tilde{u}_\epsilon = 0 & \text{on } B_0(R) \setminus \overline{B_{\rho_0}(\delta)} \cap \{x_1 = 0\} \end{cases}$$

Again from the properties of the boundary chart $\mathcal{T}$, it follows that for any $p > 1$ there exists a constant $C_p(R, \delta)$ such that

$$\begin{aligned} \int_{B_0(R) \setminus \overline{B_{\rho_0}(\delta)} \cap \{x_1 < 0\}} \left[\frac{(\tilde{u}_\epsilon)^{2^*(s_\epsilon) - 1}}{\left| \frac{\mathcal{T}(\tilde{z}_\epsilon + \alpha_\epsilon x)}{\alpha_\epsilon} \right|^{s_\epsilon}} \right]^p dx &\le C_p(R, \delta) \int_{B_0(R) \setminus \overline{B_{\rho_0}(\delta)} \cap \{x_1 < 0\}} \frac{1}{\left| \frac{\tilde{z}_\epsilon}{\alpha_\epsilon} + x \right|^{s_\epsilon p}} \, dx \\ &\le C_p(R, \delta) \int_{B_0(R)} \frac{1}{\left| \frac{\tilde{z}_\epsilon}{\alpha_\epsilon} + x \right|^{s_\epsilon p}} \, dx \end{aligned}$$

Choosing $s_\epsilon > 0$ sufficiently small it follows that the right hand side of equation (4.63) is uniformly bounded in L^p for some $p > n$. Then from standard elliptic estimates (see for instance [**14**]) we have

$$\|\tilde{u}_\epsilon\|_{C^1\left(\overline{B_0(R/2) \setminus B_{\rho_0}(2\delta) \cap \{x_1 \le 0\}}\right)} = O(1) \qquad \text{as } \epsilon \to 0$$

and $\tilde{u}_\epsilon$ vanishes on the boundary $B_0(R/2) \setminus \overline{B_{\rho_0}(2\delta)} \cap \{x_1 = 0\}$. Let $\tilde{y}_\epsilon \in U$ be such that $\mathcal{T}(\tilde{y}_\epsilon) = y_\epsilon$. From the properities of the boundary map $\mathcal{T}$ it follows that a constant $C_{\mathcal{T}} > 0$

$$\frac{1}{C_{\mathcal{T}}} \leq \frac{\tilde{y}_\epsilon - \tilde{x}_\epsilon}{|y_\epsilon - x_\epsilon|} = \left| \frac{\tilde{y}_\epsilon - \tilde{z}_\epsilon}{\alpha_\epsilon} - \rho_\epsilon \right|$$

Therefore we can choose $\delta > 0$ small and $R > 0$ large such that for $\epsilon > 0$ small enough.

$$\frac{\tilde{y}_\epsilon - \tilde{z}_\epsilon}{\alpha_\epsilon} \in B_0(R/2) \setminus \overline{B_{\rho_0}(2\delta)} \cap \{x_1 < 0\}$$

It then follows that as $\epsilon \to 0$

$$\left| \nabla \tilde{u}_\epsilon \left(\frac{\tilde{y}_\epsilon - \tilde{z}_\epsilon}{\alpha_\epsilon} \right) \right| = O(1), \qquad \tilde{u}_\epsilon \left(\frac{\tilde{y}_\epsilon - \tilde{z}_\epsilon}{\alpha_\epsilon} \right) = O(1)$$

and since $\tilde{u}_\epsilon$ vanishes on the boundary $B_0(R/2) \setminus \overline{B_{\rho_0}(2\delta)} \cap \{x_1 = 0\}$, it follows that

$$0 \leq \tilde{u}_\epsilon \left(\frac{\tilde{y}_\epsilon - \tilde{z}_\epsilon}{\alpha_\epsilon} \right) = O\left(\frac{(\tilde{y}_\epsilon - \tilde{z}_\epsilon)_1}{\alpha_\epsilon} \right) = O\left(\frac{(\tilde{y}_\epsilon)_1}{\alpha_\epsilon} \right) = O\left(\frac{d(y_\epsilon, \partial\Omega)}{\alpha_\epsilon} \right)$$

comig back to the defination of $\tilde{u}_\epsilon$ this implies that as $\epsilon \to 0$

$$|y_\epsilon - x_\epsilon|^{n/2} |\nabla u_\epsilon(y_\epsilon)| = O(1),$$
$$\frac{|y_\epsilon - x_\epsilon|^{n/2} u_\epsilon(y_\epsilon)}{d(x_\epsilon, \partial\Omega)} = O(1)$$

But this contradicts (4.61). This ends Case 2.2.

All these cases prove Proposition 4.5.2. □

As a consequence of Proposition 4.5.2, we get the following:

Corollary 4.5.1. *Let $(u_\epsilon)_{\epsilon>0}$ be as in theorem 4.4, and let $\lim_{\epsilon\to 0} x_\epsilon \to x_0 \in \overline{\Omega}$, then upto a subsequence*

$$\lim_{\epsilon \to 0} u_\epsilon = 0 \qquad \text{in } C^1_{loc}(\overline{\Omega} \backslash \{x_0\})$$

Proof. Let $\Omega' \subset\subset \overline{\Omega} \backslash \{x_0\}$ be a compactly contained open set. Then it follows from the bound obtained in proposition 4.5.1, that $\|u_\epsilon\|_{L^\infty(\Omega')} < +\infty$ for all $\epsilon > 0$. So $\dfrac{u_\epsilon^{2^*(s_\epsilon)-1}}{|x|^{s_\epsilon}} \in L^p(\Omega')$ for any $p > n$, and $\epsilon > 0$. From eqn (4.4) and with standard elliptic estimates (see for instance [**14**]) it follows that for all ϵ

$$\|u_\epsilon\|_{C^{1,\alpha}(\Omega')} \leq C'$$

for some positive constant C' and $\alpha \in (0, 1)$. Hence the sequence (u_ϵ) is precompact in the space $C^1(\overline{\Omega'})$. Since $u_\epsilon \rightharpoonup 0$ weakly in $H^2_{1,0}(\Omega)$, therefore $u_\epsilon \to 0$ in $C^1(\overline{\Omega'})$, as $\epsilon \to 0$. Note that if $0 \notin \Omega'$ then

$$\frac{u_\epsilon(x)^{\frac{2-s_\epsilon}{n-2}}}{|x|^{s_\epsilon/2}} = O\left(u_\epsilon(x)^{\frac{2-s_\epsilon}{n-2}} \right) \qquad \text{for all } x \in \Omega'$$

And if $0 \in \Omega'$ then

$$0 \leq \frac{u_\epsilon(x)^{\frac{2-s_\epsilon}{n-2}}}{|x|^{s_\epsilon/2}} \leq \left(\sup_{x\in\Omega'} |\nabla u_\epsilon|^{\frac{2-s_\epsilon}{n-2}}\right) \frac{|x|^{\frac{2-s_\epsilon}{n-2}}}{|x|^{s_\epsilon/2}} = \left(\sup_{x\in\Omega'} |\nabla u_\epsilon|^{\frac{2-s_\epsilon}{n-2}}\right) |x|^{\left(\frac{2-s_\epsilon}{n-2}-\frac{s_\epsilon}{2}\right)}$$

$$= O\left(\sup_{x\in\Omega'} |\nabla u_\epsilon|^{\frac{2-s_\epsilon}{n-2}}\right) \qquad \text{for all } x \in \Omega'$$

Since $u_\epsilon \to 0$ in $C^1(\overline{\Omega'})$ as $\epsilon \to 0$ therefore we also have that

$$\lim_{\epsilon\to 0} \frac{u_\epsilon(x)^{\frac{2-s_\epsilon}{n-2}}}{|x|^{s_\epsilon/2}} = 0 \qquad in\ C^0_{loc}(\overline{\Omega}\backslash\{x_0\})$$

□

We slightly improve our estimate in Proposition 4.5.1 to obtain

Proposition 4.5.3. *With the same hypothesis as in theorem 4.4 we have*

$$\lim_{R\to+\infty} \lim_{\epsilon\to 0} \sup_{x\in\Omega\backslash B_{x_\epsilon}(Rk_\epsilon)} |x-x_\epsilon|^{\frac{n-2}{2}} u_\epsilon(x) = 0$$

Proof. Suppose on the contrary there exists $\epsilon_0 > 0$ and a sequence of points $(y_\epsilon)_{\epsilon>0} \in \Omega$ such that upto a subsequence

$$|y_\epsilon - x_\epsilon|^{\frac{n-2}{2}} u_\epsilon(y_\epsilon) \geq \epsilon_0^{\frac{n-2}{2}} \qquad \text{and} \qquad \lim_{\epsilon\to 0} \frac{|y_\epsilon - x_\epsilon|}{k_\epsilon} = +\infty \tag{4.64}$$

It then follows from corollary 4.5.1 that

$$\lim_{\epsilon\to 0} |y_\epsilon - x_\epsilon| = 0$$

Let

$$\lambda_\epsilon^{-\frac{n-2}{2}} = u_\epsilon(y_\epsilon)$$

Then (4.64) becomes

$$C \geq \frac{|y_\epsilon - x_\epsilon|}{\lambda_\epsilon} \geq \epsilon_0 \qquad \text{for all } \ \epsilon > 0 \tag{4.65}$$

and so

$$\lim_{\epsilon\to 0} \lambda_\epsilon = 0$$

Since $\lim_{\epsilon\to 0} \frac{|y_\epsilon - x_\epsilon|}{k_\epsilon} = +\infty$, using proposition 4.5.1 we obtain that as $\epsilon \to 0$

$$\frac{k_\epsilon}{\lambda_\epsilon} = \frac{k_\epsilon}{|y_\epsilon - x_\epsilon|} \frac{|y_\epsilon - x_\epsilon|}{\lambda_\epsilon} = O\left(\frac{k_\epsilon}{|y_\epsilon - x_\epsilon|}\right) = o(1) \tag{4.66}$$

We let

$$l_\epsilon = |y_\epsilon|^{s_\epsilon/2} \lambda_\epsilon^{\frac{2-s_\epsilon}{2}} \qquad \text{for } \epsilon > 0$$

Then

$$\lim_{\epsilon\to 0} l_\epsilon = 0$$

Step 1: We claim that

$$\frac{|y_\epsilon|^{s_\epsilon}}{\lambda_\epsilon^{s_\epsilon}} = O(1) \qquad \text{as } \epsilon \to 0 \tag{4.67}$$

Proof. We proceed by contradiction. Suppose that

$$\lim_{\epsilon\to 0}\frac{\lambda_\epsilon^{s_\epsilon}}{|y_\epsilon|^{s_\epsilon}}=0$$

Now

$$\frac{|x_\epsilon|_\epsilon^{s_\epsilon}}{|y_\epsilon|^{s_\epsilon}}=\frac{\lambda_\epsilon^{s_\epsilon}}{|y_\epsilon|^{s_\epsilon}}\frac{|x_\epsilon|^{s_\epsilon}}{\lambda_\epsilon^{s_\epsilon}}\leq\frac{\lambda_\epsilon^{s_\epsilon}}{|y_\epsilon|^{s_\epsilon}}\frac{|x_\epsilon|^{s_\epsilon}}{\mu_\epsilon^{s_\epsilon}}$$

Since $\lim\limits_{\epsilon\to 0}\frac{|x_\epsilon|_\epsilon^{s_\epsilon}}{\mu_\epsilon^{s_\epsilon}}=1$, it then follows that in this case

$$\lim_{\epsilon\to 0}\frac{|x_\epsilon|^{s_\epsilon}}{|y_\epsilon|^{s_\epsilon}}=0$$

And in particular one has that $\lim\limits_{\epsilon\to 0}\frac{|x_\epsilon|}{|y_\epsilon|}=0$ and $\lim\limits_{\epsilon\to 0}\frac{\lambda_\epsilon}{|y_\epsilon|}=0$. Then

$$\frac{|y_\epsilon-x_\epsilon|}{\lambda_\epsilon}\geq\frac{|y_\epsilon|}{\lambda_\epsilon}\left|1-\frac{|x_\epsilon|}{|y_\epsilon|}\right|\to+\infty \text{ as } \epsilon\to 0$$

A contradiction to (4.65) . This completes the proof of (4.67) and ends Step 1. $\square$

Step 2: We claim that there exists $c_2>0$ such that for $\epsilon>0$ small

$$\frac{|y_\epsilon-x_\epsilon|}{l_\epsilon}=\frac{|y_\epsilon-x_\epsilon|}{\lambda_\epsilon}\frac{\lambda_\epsilon^{s_\epsilon/2}}{|y_\epsilon|^{s_\epsilon/2}}\geq c_2 \tag{4.68}$$

This follows directly from (4.67) and (4.65).

Step 3: We assume that there exists $\rho_0>0$ such that upto a subsequence

$$\frac{d(y_\epsilon,\partial\Omega)}{l_\epsilon}\geq 2\rho_0 \tag{4.69}$$

Without loss of generality we can take $2\rho_0<c_2$. For $\epsilon>0$ we let

$$w_\epsilon(x)=\lambda_\epsilon^{\frac{n-2}{2}}u_\epsilon\left(y_\epsilon+l_\epsilon x\right)\qquad\text{for }\ x\in B_0(\rho_0)$$

This is well defined since $B_{y_\epsilon}(l_\epsilon\rho_0)\subset\Omega$. Using eqn (4.4) it follows that for $\epsilon>0$ w_ϵ satisfies the equation

$$\Delta w_\epsilon+l_\epsilon^2 a\left(y_\epsilon+l_\epsilon x\right)w_\epsilon=\frac{w_\epsilon^{2^*(s_\epsilon)-1}}{\left|\frac{y_\epsilon}{|y_\epsilon|}+\frac{l_\epsilon}{|y_\epsilon|}x\right|^{s_\epsilon}}\qquad\text{in } \mathscr{D}'(B_0(\rho_0)) \tag{4.70}$$

From proposition 4.5.1 we have for some constant $C>0$

$$|l_\epsilon x+y_\epsilon-x_\epsilon|^{\frac{n-2}{2}}w_\epsilon(x)\leq C\,\lambda_\epsilon^{\frac{n-2}{2}}\qquad\text{for }\epsilon>0\quad\text{and}\ \ x\in B_0(\rho_0)$$

And so

$$\begin{aligned}
w_\epsilon(x) &\leq C\left(\frac{1}{\left|x-\left(\frac{x_\epsilon-y_\epsilon}{l_\epsilon}\right)\right|}\right)^{\frac{n-2}{2}}\left(\frac{\lambda_\epsilon}{l_\epsilon}\right)^{\frac{n-2}{2}} \qquad \text{for } \epsilon>0 \quad \text{and} \quad x\in B_0(\rho_0)\\
&\leq C\left(\frac{1}{\left|x-\left(\frac{x_\epsilon-y_\epsilon}{l_\epsilon}\right)\right|}\right)^{\frac{n-2}{2}}\left(\frac{l_\epsilon}{|y_\epsilon|}\right)^{\frac{s_\epsilon(n-2)}{2(2-s_\epsilon)}} \qquad \text{for } \epsilon>0 \quad \text{and} \quad x\in B_0(\rho_0)\\
&\leq C\left(\frac{1}{\left|x-\left(\frac{x_\epsilon-y_\epsilon}{l_\epsilon}\right)\right|}\right)^{\frac{n-2}{2}}\left(\frac{l_\epsilon}{d(y_\epsilon,\partial\Omega)}\right)^{\frac{s_\epsilon(n-2)}{2(2-s_\epsilon)}} \qquad \text{for } \epsilon>0 \quad \text{and} \quad x\in B_0(\rho_0)\\
&\leq C\left(\frac{1}{\left|x-\left(\frac{x_\epsilon-y_\epsilon}{l_\epsilon}\right)\right|}\right)^{\frac{n-2}{2}}\left(\frac{1}{2\rho_0}\right)^{\frac{s_\epsilon(n-2)}{2(2-s_\epsilon)}} \qquad \text{for } \epsilon>0 \quad \text{and} \quad x\in B_0(\rho_0)
\end{aligned}$$

And we have for $x\in B_0(\rho_0)$

$$\begin{aligned}
&c_2 \leq \frac{|y_\epsilon-x_\epsilon|}{l_\epsilon} \leq \left|x-\left(\frac{x_\epsilon-y_\epsilon}{l_\epsilon}\right)\right|+\rho_0 \leq \left|x-\left(\frac{x_\epsilon-y_\epsilon}{l_\epsilon}\right)\right|+\frac{c_2}{2},\\
&\Longrightarrow \frac{c_2}{2} \leq \left|x-\left(\frac{x_\epsilon-y_\epsilon}{\lambda_\epsilon}\right)\right|
\end{aligned}$$

And so there exists a constant $C_0>0$ such that

$$w_\epsilon(x)\leq C_0 \qquad \text{for } \epsilon>0 \quad \text{and} \quad x\in B_0(\rho_0)$$

Also we have in this case : $\frac{|y_\epsilon|}{l_\epsilon} \geq \frac{d(y_\epsilon,\partial\Omega)}{l_\epsilon} \geq 2\rho_0$, so $\frac{l_\epsilon}{|y_\epsilon|} \leq \frac{1}{2\rho_0}$, and therefore for $x\in B_0(\rho_0)$

$$\frac{1}{2} \leq \left|\frac{y_\epsilon}{|y_\epsilon|}+\frac{l_\epsilon}{|y_\epsilon|}x\right| \leq \frac{3}{2}$$

Coming back to equation (4.70) we then have that the right side of the equation is uniformly bounded in L^∞ for $\epsilon>0$ small. Then, again by standard elliptic estimates it follows that that there exists $w_0\in C^1\left(B_0(\rho_0)\right)$ such that up to a subsequence

$$\lim_{\epsilon\to 0} w_\epsilon = w_0 \qquad \text{in } C^1\left(B_0(\rho_0/2)\right)$$

So in particular $w_0(0) = 1$. We have for $\epsilon > 0$

$$\int\limits_{B_{y_\epsilon}(\frac{\rho_0}{2}l_\epsilon)} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\,dx = \left(\frac{|y_\epsilon|^{s_\epsilon}}{\lambda_\epsilon^{s_\epsilon}}\right)^{\frac{n-2}{2}} \int\limits_{B_0(\frac{\rho_0}{2})} \frac{|w_\epsilon(x)|^{2^*(s_\epsilon)}}{\left|\frac{y_\epsilon}{|y_\epsilon|}+\frac{l_\epsilon}{|y_\epsilon|}x\right|^{s_\epsilon}}\,dx$$

$$= \left(\frac{|y_\epsilon|}{l_\epsilon}\right)^{\frac{s_\epsilon(n-2)}{2(2-s_\epsilon)}} \int\limits_{B_0(\frac{\rho_0}{2})} \frac{|w_\epsilon(x)|^{2^*(s_\epsilon)}}{\left|\frac{y_\epsilon}{|y_\epsilon|}+\frac{l_\epsilon}{|y_\epsilon|}x\right|^{s_\epsilon}}\,dx$$

$$\geq \left(\frac{d(y_\epsilon,\partial\Omega)}{l_\epsilon}\right)^{\frac{s_\epsilon(n-2)}{2(2-s_\epsilon)}} \int\limits_{B_0(\frac{\rho_0}{2})} \frac{|w_\epsilon(x)|^{2^*(s_\epsilon)}}{\left|\frac{y_\epsilon}{|y_\epsilon|}+\frac{l_\epsilon}{|y_\epsilon|}x\right|^{s_\epsilon}}\,dx$$

$$\geq (2\rho_0)^{\frac{s_\epsilon(n-2)}{2(2-s_\epsilon)}} \int\limits_{B_0(\frac{\rho_0}{2})} \frac{|w_\epsilon(x)|^{2^*(s_\epsilon)}}{\left|\frac{y_\epsilon}{|y_\epsilon|}+\frac{l_\epsilon}{|y_\epsilon|}x\right|^{s_\epsilon}}\,dx$$

Passing to the limit as $\epsilon \to 0$, we have

$$\lim_{\epsilon\to 0} \int\limits_{B_{y_\epsilon}(\frac{\rho_0}{2}l_\epsilon)} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\,dx \geq \int\limits_{B_0(\frac{\rho_0}{2})} w_0^{2^*}\,dx$$

We have shown in proposition 4.4.1 that

$$\lim_{R\to+\infty}\lim_{\epsilon\to 0} \int\limits_{\Omega\setminus B_{x_\epsilon}(Rk_\epsilon)} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\,dx = 0$$

So given any $\tilde{\delta} > 0$, there exists $\tilde{R}$ large, and $\tilde{\epsilon} > 0$ small such that

$$\int\limits_{\Omega\setminus B_{x_\epsilon}(\tilde{R}k_\epsilon)} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\,dx \leq \tilde{\delta} \qquad \text{for } \epsilon < \tilde{\epsilon}$$

From (4.68) it follows that for $\epsilon > 0$ small

$$2\rho_0 < c_2 \leq \frac{|y_\epsilon - x_\epsilon|}{l_\epsilon} \leq \frac{|x_\epsilon - x|}{l_\epsilon} + \frac{\rho_0}{2} \qquad \text{for } x \in B_{y_\epsilon}\left(\frac{\rho_0}{2}l_\epsilon\right)$$

Therefore for $\epsilon > 0$ small

$$\frac{|x_\epsilon - x|}{k_\epsilon} \geq \frac{3c_2}{4} \qquad \text{for } x \in B_{y_\epsilon}\left(\frac{\rho_0}{2}l_\epsilon\right)$$

Using (4.66) we have that

$$\frac{k_\epsilon}{l_\epsilon} = \frac{k_\epsilon}{\lambda_\epsilon}\frac{\lambda_\epsilon}{l_\epsilon} = \frac{k_\epsilon}{\lambda_\epsilon}\frac{\lambda_\epsilon}{l_\epsilon} \leq \left[\frac{k_\epsilon}{\lambda_\epsilon}\left(\frac{1}{2\rho_0}\right)^{\frac{s_\epsilon}{2-s_\epsilon}}\right]$$

Therefore there exists $\tilde{\epsilon'} > 0$ small such that for $\epsilon < \tilde{\epsilon'}$

$$\frac{|x_\epsilon - x|}{k_\epsilon} \geq \frac{3c_2}{4}\frac{l_\epsilon}{k_\epsilon} \geq \frac{3c_2}{4}\tilde{R} \qquad \text{for } x \in B_{y_\epsilon}\left(\frac{\rho_0}{2}l_\epsilon\right)$$

So for $\epsilon < \tilde{\epsilon'}$

$$\Omega \cap B_{y_\epsilon}\left(\frac{\rho_0}{2}l_\epsilon\right) \subset \Omega\setminus B_{x_\epsilon}(\tilde{R}k_\epsilon)$$

And hence for all $\delta' > 0$

$$\int_{B_{y_\epsilon}(\frac{\rho_0}{2}l_\epsilon)} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\,dx \leq \int_{\Omega\backslash B_{x_\epsilon}(\tilde{R}k_\epsilon)} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\,dx \leq \tilde{\delta} \qquad \text{for } \epsilon < \min\{\tilde{\epsilon}, \tilde{\epsilon'}\}$$

Since $\tilde{\delta} > 0$ is arbitrary, it follows that

$$\int_{B_0(\frac{\rho_0}{2})} w_0^{2^*}\,dx \leq \lim_{\epsilon\to 0} \int_{B_{y_\epsilon}(\frac{\rho_0}{2}l_\epsilon)} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\,dx \leq \lim_{\epsilon\to 0} \int_{\Omega\backslash B_{x_\epsilon}(\tilde{R}k_\epsilon)} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\,dx = 0$$

and then $w_0 \equiv 0$ in $B_0(\rho_0/2)$, a contradiction since we have earlier obtained that $w_0(0) = 1$. This proves proposition 4.5.3 when (4.69) holds, and therefore this ends Step 3.

Step 4: We assume that

$$\lim_{\epsilon\to 0} \frac{d(y_\epsilon, \partial\Omega)}{l_\epsilon} = 0 \tag{4.71}$$

We note that then one also has from (4.67)

$$\frac{d(y_\epsilon, \partial\Omega)}{\lambda_\epsilon} = \frac{|y_\epsilon|^{s_\epsilon}}{\lambda_\epsilon^{s_\epsilon}}\frac{d(y_\epsilon, \partial\Omega)}{l_\epsilon} = o(1) \qquad \text{as } \epsilon \to 0$$

and

$$\lim_{\epsilon\to 0} y_\epsilon = y_0 \in \partial\Omega$$

Let $\mathcal{T}$ be a parametrisation of the boundary $\partial\Omega$ as in (4.18) around the point $p = y_0$ For all $\epsilon > 0$ let

$$\tilde{u}_\epsilon = u_\epsilon \circ \mathcal{T} \qquad \text{on } U \cap \{x_1 \leq 0\}$$

For $i, j = 1, \ldots, n$, let $g_{ij} = (\partial_i\mathcal{T}, \partial_j\mathcal{T})$ be the metric induced by the chart $\mathcal{T}$ on the domain $U \cap \{x_1 < 0\}$, and let Δ_g denote the Laplace-Beltrami operator with respect to the metric g. From equation (4.4) it follows that for any $\epsilon > 0$, $\tilde{u}_\epsilon$ satisfies weakly the equation

$$\begin{cases} \Delta_g \tilde{u}_\epsilon + a \circ \mathcal{T}(x)\tilde{u}_\epsilon = \frac{\tilde{u}_\epsilon^{2^*(s_\epsilon)-1}}{|\mathcal{T}(x)|^{s_\epsilon}} & \text{in } U \cap \{x_1 < 0\} \\ \tilde{u}_\epsilon = 0 & \text{on } U \cap \{x_1 = 0\} \end{cases} \tag{4.72}$$

Let $z'_\epsilon \in \partial\Omega$ be such that

$$|z'_\epsilon - y_\epsilon| = d(y_\epsilon, \partial\Omega) \qquad \text{for } \epsilon > 0$$

And let $\tilde{y}_\epsilon$, $\tilde{z}'_\epsilon \in U$ be such that

$$\mathcal{T}(\tilde{y}_\epsilon) = y_\epsilon \qquad \text{and} \qquad \mathcal{T}(\tilde{z}'_\epsilon) = z'_\epsilon$$

Then it follows from the properties of the boundary chart $\mathcal{T}$, that

$$\lim_{\epsilon\to 0} \tilde{y}_\epsilon = 0 = \lim_{\epsilon\to 0} \tilde{z}'_\epsilon\ , \qquad (\tilde{y}_\epsilon)_1 < 0 \ \text{ and } (\tilde{z}'_\epsilon)_1 = 0$$

We let

$$\tilde{w}_\epsilon = \frac{\tilde{u}_\epsilon\,(\tilde{z}'_\epsilon + \lambda_\epsilon x)}{\tilde{u}_\epsilon(\tilde{y}_\epsilon)} \qquad \text{for } x \in B_0(\epsilon_0/4) \cap \{x_1 \leq 0\}$$

$\tilde{w}_\epsilon$ is well defined for $\epsilon > 0$ small sufficiently enough. Let

$$\tilde{g}_\epsilon = g\left(\tilde{z}'_\epsilon + \lambda_\epsilon x\right)$$

Then for ϵ sufficiently small, $\tilde{w}_\epsilon$ satisfies weakly the equation

$$(4.73)\quad \begin{cases} \Delta_{\tilde{g}_\epsilon}\tilde{w}_\epsilon + \lambda_\epsilon^2\left(a\circ\mathcal{T}\left(\tilde{z}'_\epsilon + \lambda_\epsilon x\right)\right)\tilde{w}_\epsilon = \dfrac{\tilde{w}_\epsilon^{2^*(s_\epsilon)-1}}{\left|\frac{\mathcal{T}(\tilde{z}'_\epsilon+\lambda_\epsilon x)}{\lambda_\epsilon}\right|^{s_\epsilon}} & \text{in } B_0(\epsilon_0/4)\cap\{x_1<0\} \\ \tilde{w}_\epsilon = 0 & \text{on } B_0(\epsilon_0/4)\cap\{x_1=0\} \end{cases}$$

From proposition 4.5.1 we have for a constant C

$$\left|\mathcal{T}\left(\tilde{z}'_\epsilon + \lambda_\epsilon x\right) - x_\epsilon\right|^{\frac{n-2}{2}}\tilde{w}_\epsilon(x) \le C\lambda_\epsilon^{\frac{n-2}{2}},$$

$$\left(\frac{\left|\mathcal{T}\left(\tilde{z}'_\epsilon + \lambda_\epsilon x\right) - x_\epsilon\right|}{\lambda_\epsilon}\right)^{\frac{n-2}{2}}\tilde{w}_\epsilon(x) \le C$$

It follows from the properties of the map $\mathcal{T}_0$, that for $\epsilon > 0$ sufficiently small

$$\frac{\left|\mathcal{T}\left(\tilde{z}'_\epsilon + \lambda_\epsilon x\right) - x_\epsilon\right|}{\lambda_\epsilon} \ge \frac{\epsilon_0}{2} \qquad \text{for } x \in B_0(\epsilon_0/4)\cap\{x_1\le 0\}$$

So there exists a constant $C_0 > 0$ such that for $\epsilon > 0$ sufficiently small we have

$$\tilde{w}_\epsilon(x) \le C_0 \qquad \text{for } x \in B_0(\epsilon_0/4)\cap\{x_1\le 0\}$$

Again from the properties of the boundary chart $\mathcal{T}$, it follows that for any $p > 1$ there exists a constant C_p such that

$$\int_{B_0(\epsilon_0/4)\cap\{x_1<0\}}\left[\frac{\left(\tilde{w}_\epsilon\right)^{2^*(s_\epsilon)-1}}{\left|\frac{\mathcal{T}(\tilde{z}'_\epsilon+\lambda_\epsilon x)}{\lambda_\epsilon}\right|^{s_\epsilon}}\right]^p dx \le C_p\int_{B_0(\epsilon_0/4)\cap\{x_1<0\}}\frac{1}{\left|\frac{\tilde{z}'_\epsilon}{\lambda_\epsilon}+x\right|^{s_\epsilon p}}\,dx$$

Choosing $s_\epsilon > 0$ sufficiently small it follows that the right hand side of equation (4.73) is uniformly bounded in L^p for some $p > n$. Then from standard elliptic estimates (see for instance [**14**]) we have that, there exists $\tilde{w} \in C^1\left(B_0(\epsilon_0/8)\cap\{x_1 \leqq 0\}\right)$ such that up to a subsequence

$$\lim_{\epsilon\to 0}\tilde{w}_\epsilon = \tilde{w} \qquad \text{in } C^1\left(B_0(\epsilon_0/16)\cap\{x_1\le 0\}\right)$$

And therefore, in particular

$$(4.74)\qquad \tilde{w} \equiv 0 \qquad \text{on } B_0(\epsilon_0/16)\cap\{x_1 = 0\}$$

One has for all $\epsilon > 0$

$$\tilde{w}_\epsilon\left(\frac{\tilde{y}_\epsilon - \tilde{z}'_\epsilon}{\lambda_\epsilon}\right) = 1$$

And from the properties of the boundary chart $\mathcal{T}$ it follows that, for all $\epsilon > 0$ in this case

$$\frac{|\tilde{y}_\epsilon - \tilde{z}'_\epsilon|}{\lambda_\epsilon} = O\left(\frac{|y_\epsilon - z'{}_\epsilon|}{\lambda_\epsilon}\right) = O\left(\frac{d(y_\epsilon,\partial\Omega)}{\lambda_\epsilon}\right) = o\left(\frac{l_\epsilon}{\lambda_\epsilon}\right) = o(1)$$

As , $\lim_{\epsilon\to 0}\tilde{w}_\epsilon = \tilde{w}$ in $C^1\left(B_0(\epsilon_0/16)\cap\{x_1\le 0\}\right)$ then $\tilde{w}(0) = 1$. But this contradicts what we have obtained in (4.74), proving proposition 4.5.3 when (4.71) holds, and ends Step 4.

These four steps complete the proof of proposition 4.5.3. $\square$

4.6. Refined Blowup Analysis II

Now we proceed to prove the main theorem of this section.

Theorem 4.6. *Let Ω be a bounded smooth oriented domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$, and let $a \in C^1(\overline{\Omega})$ be such that the operator $\Delta + a$ is coercive in Ω. Let $(s_\epsilon)_{\epsilon>0} \in (0,2)$ be a sequence such that $\lim_{\epsilon\to 0} s_\epsilon = 0$. Suppose that the sequence $(u_\epsilon)_{\epsilon>0} \in H^2_{1,0}(\Omega)$, where for each $\epsilon > 0$, u_ϵ satisfies (4.4) and (4.5), is a* blowup sequence, *i.e*

$$u_\epsilon \rightharpoonup 0 \qquad \textit{weakly in } H^2_{1,0}(\Omega) \qquad \textit{as } \ \epsilon \to 0$$

Then, there exists $C > 0$ such that for all $\epsilon > 0$

$$u_\epsilon(x) \leq C\left(\frac{\mu_\epsilon}{\mu_\epsilon^2 + |x-x_\epsilon|^2}\right)^{\frac{n-2}{2}} \qquad \textit{for all } x \in \Omega$$

where

$$\mu_\epsilon^{-\frac{n-2}{2}} = u_\epsilon(x_\epsilon) = \max_{x\in\Omega} u_\epsilon(x).$$

Proof. **Step 1:** we claim that for any $\alpha \in (0, n-2)$, there exists $C_\alpha > 0$ such that for all $\epsilon > 0$

$$|x-x_\epsilon|^\alpha \, \mu_\epsilon^{\frac{n-2}{2}-\alpha} u_\epsilon(x) \leq C_\alpha \qquad \text{for all } x \in \Omega \tag{4.75}$$

Proof. Since the operator $\Delta + a$ is coercive on Ω and $a \in C(\overline{\Omega})$, there exists $U_0 \subset \mathbb{R}^n$ an open set such that $\overline{\Omega} \subset\subset U_0$, and there exists $a_1 > 0$, $A_1 > 0$ such that

$$\int_{U_0} |\nabla\varphi|^2 \ dx + \int_{U_0} (a-a_1)\,\varphi^2 \ dx \geq A_1 \int_{U_0} \varphi^2 \ dx \qquad \text{for all } \varphi \in C_c^\infty(U_0)$$

In other words the operator $\Delta + (a - a_1)$ is coercive on U_0. Here, we have extended a by 0 outside Ω (the resulting function is not necessarily continuous on $\mathbb{R}^n$).

Let $\tilde{G} : \overline{U_0}\times\overline{U_0}\backslash\{(x,x) : x \in \overline{U_0}\} \longrightarrow \mathbb{R}$ be the *Green's function* of the operator $\Delta+(a-a_1)$ with Dirichlet boundary conditions. $\tilde{G}$ satisfies in the sense of distributions

$$\Delta\tilde{G}(x,\cdot) + (a-a_1)\tilde{G}(x,\cdot) = \delta_x \tag{4.76}$$

Since the operator $\Delta + (a - a_1)$ is coercive on U_0, $\tilde{G}$ exists. See Robert [**17**].

We set for all $\epsilon > 0$

$$\tilde{G}_\epsilon(x) = \tilde{G}(x_\epsilon, x) \qquad \text{for } x \in \overline{U_0}\backslash\{x_\epsilon\} \tag{4.77}$$

$\tilde{G}_\epsilon$ satisfies for all $\epsilon > 0$

$$0 < \tilde{G}_\epsilon(x) < \frac{C}{|x-x_\epsilon|^{n-2}} \qquad \text{for } x \in \overline{U_0}\backslash\{x_\epsilon\}$$

here C is a constant. Moreover there exists $\delta_0 > 0$ and $C_0 > 0$ such that for all $\epsilon > 0$

(4.78)

$$\tilde{G}_\epsilon(x) > \frac{C_0}{|x-x_\epsilon|^{n-2}} \qquad \text{and} \qquad \frac{|\nabla\tilde{G}_\epsilon(x)|}{|\tilde{G}_\epsilon(x)|} > \frac{C_0}{|x-x_\epsilon|} \qquad \text{for } x \in B_{x_\epsilon}(\delta_0)\backslash\{x_\epsilon\} \subset\subset U_0$$

We define the operator

$$\mathcal{L}_\epsilon = \Delta + a - \frac{u_\epsilon^{2^*(s_\epsilon)-2}}{|x|^{s_\epsilon}}$$

Step 1.1: We claim that there exists $\nu_0 \in (0,1)$ such that given any $\nu \in (0,\nu_0)$ there exists $R_1 > 0$ such that for $R > R_1$ and $\epsilon > 0$ sufficiently small we have

$$\mathcal{L}_\epsilon \tilde{G}_\epsilon^{1-\nu} > 0 \qquad \text{in } \Omega \backslash B_{x_\epsilon}(Rk_\epsilon) \tag{4.79}$$

We prove the claim. We choose $\nu_0 \in (0,1)$ such that for any $\nu \in (0,\nu_0)$ one has

$$\nu\,(a - a_1) \geq -\frac{a_1}{2} \qquad \text{in } \Omega$$

Fix $\nu \in (0,\nu_0)$. We have for all $\epsilon > 0$ sufficiently small

$$\frac{\mathcal{L}_\epsilon \tilde{G}_\epsilon^{1-\nu}}{\tilde{G}_\epsilon^{1-\nu}} = (1-\nu)\frac{\Delta \tilde{G}_\epsilon}{\tilde{G}_\epsilon} + a - \frac{u_\epsilon^{2^*(s_\epsilon)-2}}{|x|^{s_\epsilon}} + \nu(1-\nu)\frac{|\nabla \tilde{G}_\epsilon|^2}{|\tilde{G}_\epsilon|^2} \qquad \text{in } \Omega\backslash\{x_\epsilon\}$$

Using (4.76) we then obtain

$$\begin{aligned}\frac{\mathcal{L}_\epsilon \tilde{G}_\epsilon^{1-\nu}}{\tilde{G}_\epsilon^{1-\nu}} =& a_1 + \nu(a-a_1) + \nu(1-\nu)\frac{|\nabla \tilde{G}_\epsilon|^2}{|\tilde{G}_\epsilon|^2} - \frac{u_\epsilon^{2^*(s_\epsilon)-2}}{|x|^{s_\epsilon}} \qquad &\text{in } \Omega\backslash\{x_\epsilon\}\\ \geq& \frac{a_1}{2} + \nu(1-\nu)\frac{|\nabla \tilde{G}_\epsilon|^2}{|\tilde{G}_\epsilon|^2} - \frac{u_\epsilon^{2^*(s_\epsilon)-2}}{|x|^{s_\epsilon}} \qquad &\text{in } \Omega\backslash\{x_\epsilon\}\end{aligned}$$

Let $|x - x_\epsilon| \geq \delta_0$,where δ_0 is as in (4.78), then from corollary 4.5.1 we have

$$\lim_{\epsilon\to 0} \frac{u_\epsilon^{2^*(s_\epsilon)-2}}{|x|^{s_\epsilon}} = 0 \qquad \text{in } C(\overline{\Omega\backslash B_{x_\epsilon}(\delta_0)})$$

Hence for $\epsilon > 0$ sufficiently small we have for $\nu \in (0,\nu_0)$

$$\frac{\mathcal{L}_\epsilon G_\epsilon^{1-\nu}}{G_\epsilon^{1-\nu}} > 0 \qquad \text{for } x \in \Omega\backslash B_{x_\epsilon}(\delta_0)$$

By strong pointwise estimates, proposition 4.5.3 we have that, given any $\nu \in (0,\nu_0)$, there exists $R_1 > 0$ such that for any $R > R_1$

$$\sup_{\Omega\backslash B_{x_\epsilon}(Rk_\epsilon)} |x - x_\epsilon|^{\frac{n-2}{2}}\, u_\epsilon(x) \leq \left[\frac{\nu(1-\nu)}{4}C_0^2\right]^{\frac{n-2}{4}}$$

Here C_0 is as in (4.78). And then using proposition 4.5.2 we obtain for $\epsilon > 0$ small

$$\begin{aligned}\sup_{\Omega\backslash B_{x_\epsilon}(Rk_\epsilon)} \frac{u_\epsilon^{2^*(s_\epsilon)-2}}{|x|^{s_\epsilon}} &= \sup_{\Omega\backslash B_{x_\epsilon}(Rk_\epsilon)} \left[u_\epsilon^{2^*(s_\epsilon)-2-s_\epsilon}\left(\frac{u_\epsilon}{|x|}\right)^{s_\epsilon}\right]\\ &\leq \left(\frac{\nu(1-\nu)}{4}C_0^2\right)^{\frac{4-ns_\epsilon}{2}} \frac{1}{|x-x_\epsilon|^{\frac{4-ns_\epsilon}{2}}}\frac{C^{s_\epsilon}}{|x-x_\epsilon|^{ns_\epsilon/2}}\\ &\leq \frac{\nu(1-\nu)}{2}\frac{C_0^2}{|x-x_\epsilon|^2}\end{aligned}$$

Therefore if $x \in B_{x_\epsilon}(\delta_0)\backslash B_{x_\epsilon}(Rk_\epsilon)$ then with the help of (4.78) we obtain for $\epsilon > 0$ small

$$\frac{\mathcal{L}_\epsilon G_\epsilon^{1-\nu}}{G_\epsilon^{1-\nu}} \geq \frac{a_1}{2} + \frac{\nu(1-\nu)}{2}\frac{C_0^2}{|x-x_\epsilon|^2} > 0$$

This proves the claim and ends Step 1.1. Hence our claim follows.

Step 1.2: Let $\nu \in (0, \nu_0)$ and $R > R_1$. We claim that there exists $C(R) > 0$ such that for $\epsilon > 0$ small

$$\mathcal{L}_\epsilon\left(C(R)\mu_\epsilon^{\frac{n-2}{2}-\nu(n-2)}G_\epsilon^{1-\nu}\right) > \mathcal{L}_\epsilon u_\epsilon \qquad in\ \Omega\backslash B_{x_\epsilon}(Rk_\epsilon)$$

$$C(R)\mu_\epsilon^{\frac{n-2}{2}-\nu(n-2)}G_\epsilon^{1-\nu} > u_\epsilon \qquad on\ \partial\left(\Omega\backslash B_{x_\epsilon}(Rk_\epsilon)\right) \tag{4.80}$$

We prove the claim. Since $\mathcal{L}_\epsilon u_\epsilon = 0$ in Ω, so it follows from (4.79) that

$$\mathcal{L}_\epsilon\left(C(R)\mu_\epsilon^{\frac{n-2}{2}-\nu(n-2)}G_\epsilon^{1-\nu}\right) > \mathcal{L}_\epsilon u_\epsilon$$

in $\Omega\backslash B_{x_\epsilon}(Rk_\epsilon)$ for $R > R_1$ and $\epsilon > 0$ sufficiently small. With (4.78) we obtain for $\epsilon > 0$ small

$$\begin{aligned}
\frac{u_\epsilon(x)}{\mu_\epsilon^{\frac{n-2}{2}-\nu(n-2)}G_\epsilon^{1-\nu}(x)} &\leq \frac{\mu_\epsilon^{-\frac{n-2}{2}}}{\mu_\epsilon^{\frac{n-2}{2}-\nu(n-2)}}\frac{(Rk_\epsilon)^{(n-2)(1-\nu)}}{C_0^{1-\nu}} \qquad \text{for } x \in \Omega\cap\partial B_{x_\epsilon}(Rk_\epsilon)\\
&= \left(\frac{|x_\epsilon|}{\mu_\epsilon}\right)^{s_\epsilon\frac{(n-2)(1-\nu)}{2}}\frac{R^{(n-2)(1-\nu)}}{C_0^{1-\nu}}\\
&\leq \frac{(2R)^{(n-2)(1-\nu)}}{C_0^{1-\nu}} \qquad \text{since } \lim_{\epsilon\to 0}\left(\frac{\mu_\epsilon}{|x_\epsilon|}\right)^{s_\epsilon} = 1
\end{aligned}$$

So for $x \in \partial\left(\Omega\backslash B_{x_\epsilon}(Rk_\epsilon)\right)$ one has for $\epsilon > 0$ small

$$\frac{u_\epsilon(x)}{\mu_\epsilon^{\frac{n-2}{2}-\nu(n-2)}G_\epsilon^{1-\gamma}(x)} \leq C(R) \qquad \text{for } x \in \Omega\cap\partial B_{x_\epsilon}(Rk_\epsilon)$$

This proves the claim and ends Step 1.2.

Step 1.3: Let $\nu \in (0, \nu_0)$ and $R > R_1$. Since $G_\epsilon^{1-\nu} > 0$ in $\overline{\Omega\backslash B_{x_\epsilon}(Rk_\epsilon)}$ and $\mathcal{L}_\epsilon G_\epsilon^{1-\nu} > 0$ in $\Omega\backslash B_{x_\epsilon}(Rk_\epsilon)$, it follows from [**3**] that the operator $\mathcal{L}_\epsilon$ satisfies the comparison principle. Then from (4.80) we have that for $\epsilon > 0$ small

$$u_\epsilon(x) \leq C(R)\mu_\epsilon^{\frac{n-2}{2}-\nu(n-2)}G_\epsilon^{1-\nu}(x) \qquad \text{for } \ x \in \Omega\backslash B_{x_\epsilon}(Rk_\epsilon)$$

Then with (4.77) we get that

$$|x-x_\epsilon|^{(n-2)(1-\nu)}u_\epsilon(x) \leq C(R)\mu_\epsilon^{\frac{n-2}{2}-\nu(n-2)} \qquad \text{for } \ x \in \Omega\backslash B_{x_\epsilon}(Rk_\epsilon)$$

Taking $\alpha = (n-2)(1-\nu)$, we have for α close to $n-2$

$$|x-x_\epsilon|^\alpha \mu_\epsilon^{\frac{n-2}{2}-\alpha}u_\epsilon(x) \leq C_\alpha \qquad \text{for } \ x \in \Omega\backslash B_{x_\epsilon}(Rk_\epsilon)$$

Let $\alpha' \in (0, \alpha)$, then

$$\begin{aligned}
|x-x_\epsilon|^{\alpha'}\mu_\epsilon^{\frac{n-2}{2}-\alpha'}u_\epsilon(x) &= \left(\frac{\mu_\epsilon}{|x-x_\epsilon|}\right)^{\alpha-\alpha'}|x-x_\epsilon|^\alpha\mu_\epsilon^{\frac{n-2}{2}-\alpha}u_\epsilon(x)\\
&\leq \left(\left(\frac{\mu_\epsilon}{|x_\epsilon|}\right)^{s_\epsilon/2}\frac{1}{R}\right)^{\alpha-\alpha'}|x-x_\epsilon|^\alpha\mu_\epsilon^{\frac{n-2}{2}-\alpha}u_\epsilon(x) \qquad \text{for } \ x \in \Omega\backslash B_{x_\epsilon}(Rk_\epsilon)
\end{aligned}$$

Where we have used (4.37). Hence for all $\alpha \in (0, n-2)$ we have that

$$|x - x_\epsilon|^\alpha \mu_\epsilon^{\frac{n-2}{2}-\alpha} u_\epsilon(x) \le C_\alpha \qquad \text{for } \; x \in \Omega \backslash B_{x_\epsilon}(Rk_\epsilon)$$

Obviously one has for $\alpha \in (0, n-2)$

$$|x - x_\epsilon|^\alpha \mu_\epsilon^{\frac{n-2}{2}-\alpha} u_\epsilon(x) \le C_\alpha \qquad \text{for } \; x \in B_{x_\epsilon}(Rk_\epsilon)$$

These two inequalities prove (4.75). This ends Step 1.3 and also Step 1. □

Next we show that one can infact take $\alpha = n-2$ in (4.75).

Step 2: We claim that there exists $C > 0$ such that for all $\epsilon > 0$

$$|x - x_\epsilon|^{n-2} u_\epsilon(x_\epsilon) \; u_\epsilon(x) \le C \qquad \text{for all } x \in \Omega \tag{4.81}$$

PROOF. Let $y_\epsilon \in \Omega$ be such that

$$|y_\epsilon - x_\epsilon|^{n-2} u_\epsilon(x_\epsilon) \; u_\epsilon(y_\epsilon) = \sup_{x \in \Omega} |x - x_\epsilon|^{n-2} u_\epsilon(x_\epsilon) \; u_\epsilon(x)$$

Then (4.81) is equivalent to proving that

$$|y_\epsilon - x_\epsilon|^{n-2} u_\epsilon(x_\epsilon) \; u_\epsilon(y_\epsilon) = O(1) \qquad as \; \epsilon \to 0$$

We have the following two cases.

Step 2.1: Suppose that

$$|x_\epsilon - y_\epsilon| = O(\mu_\epsilon) \qquad as \; \epsilon \to 0$$

By definition (4.14) it follows that

$$|y_\epsilon - x_\epsilon|^{n-2} u_\epsilon(x_\epsilon) \; u_\epsilon(y_\epsilon) \le |y_\epsilon - x_\epsilon|^{n-2} \mu_\epsilon^{2-n}$$

This proves (4.81) in this case and ends Step 2.1.

Step 2.2: Suppose that

$$\lim_{\epsilon \to 0} \frac{|x_\epsilon - y_\epsilon|}{\mu_\epsilon} = +\infty \qquad as \; \epsilon \to 0$$

We let for $\epsilon > 0$

$$\hat{v}_\epsilon(x) = \mu_\epsilon^{\frac{n-2}{2}} u_\epsilon \left(\mu_\epsilon x + x_\epsilon\right) \qquad \text{for } \; x \in \frac{\Omega - x_\epsilon}{\mu_\epsilon}$$

Then from (4.75), it follows that for any $\alpha \in (0, n-2)$, there exists $C_\alpha > 0$ such that for all $\epsilon > 0$

$$\begin{aligned} |\mu_\epsilon x|^\alpha \mu_\epsilon^{\frac{n-2}{2}-\alpha} u_\epsilon \left(\mu_\epsilon x + x_\epsilon\right) &\le C_\alpha \qquad \text{for } x \in \Omega, \\ |x|^\alpha \hat{v}_\epsilon(x) &\le C_\alpha \qquad \text{for } \; x \in \frac{\Omega - x_\epsilon}{\mu_\epsilon} \end{aligned}$$

And so

$$\hat{v}_\epsilon(x) + |x|^\alpha \hat{v}_\epsilon(x) \le 1 + C_\alpha = C'_\alpha \qquad \text{for } \; x \in \frac{\Omega - x_\epsilon}{\mu_\epsilon}$$

Hence for all $\epsilon > 0$ and $\alpha \in (0, n-2)$, we have for a constant $C'_\alpha > 0$

$$\hat{v}_\epsilon(x) \le \frac{C'_\alpha}{1 + |x|^\alpha} \qquad \text{for } \; x \in \frac{\Omega - x_\epsilon}{\mu_\epsilon}$$

Let $G \in C^2(\Omega \times \Omega \backslash \{(x,x) : x \in \Omega\})$ be the *Green's function* of the coercive operator $\Delta + a$ with Dirichlet boundary conditions on Ω. It follows from *Green's representation formula* that

$$u_\epsilon(y_\epsilon) = \int_\Omega G(x, y_\epsilon) \frac{u_\epsilon^{2^*(s_\epsilon)-1}(x)}{|x|^{s_\epsilon}}\, dx \qquad \text{for all } \epsilon > 0$$

using the estimates on Green's function this becomes

$$u_\epsilon(y_\epsilon) \leq C \int_\Omega \frac{1}{|x-y_\epsilon|^{n-2}} \frac{u_\epsilon^{2^*(s_\epsilon)-1}(x)}{|x|^{s_\epsilon}}\, dx \qquad \text{for all } \epsilon > 0 \tag{4.82}$$

where $C > 0$ is a constant. We write the above integral as follows

$$u_\epsilon(y_\epsilon) \leq C \int_\Omega \left(\frac{u_\epsilon(x)}{|x|}\right)^{s_\epsilon} \frac{1}{|x-y_\epsilon|^{n-2}}\, u_\epsilon(x)^{2^*(s_\epsilon)-1-s_\epsilon}\, dx \qquad \text{for all } \epsilon > 0$$

Using Hölder inequality and then by Hardy inequality (4.8) we get for all $\epsilon > 0$

$$\begin{aligned} u_\epsilon(y_\epsilon) \leq & C \left(\int_\Omega \frac{|u_\epsilon(x)|^2}{|x|^2}\, dx \right)^{s_\epsilon/2} \left(\int_\Omega \left(\frac{1}{|x-y_\epsilon|^{n-2}}\right)^{\frac{2}{2-s_\epsilon}} u_\epsilon(x)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{2}{2-s_\epsilon}}\, dx \right)^{\frac{2-s_\epsilon}{2}} \\ \leq & C \left(\left(\frac{2}{n-2}\right)^2 \int_\Omega |\nabla u_\epsilon|^2\, dx \right)^{s_\epsilon/2} \left(\int_\Omega \left(\frac{1}{|x-y_\epsilon|^{n-2}}\right)^{\frac{2}{2-s_\epsilon}} u_\epsilon(x)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{2}{2-s_\epsilon}}\, dx \right)^{\frac{2-s_\epsilon}{2}} \quad \text{for a} \end{aligned}$$

The sequence $(u_\epsilon)_{\epsilon>0}$ is bounded in $H^2_{1,0}(\Omega)$ as shown in (4.11), so it follows that there exists a constant $C > 0$ such that for $\epsilon > 0$ small

$$u_\epsilon(y_\epsilon)^{\frac{2}{2-s_\epsilon}} \leq C \int_\Omega \frac{1}{|x-y_\epsilon|^{\frac{2(n-2)}{2-s_\epsilon}}}\, u_\epsilon(x)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{2}{2-s_\epsilon}}\, dx$$

With a change of variables the above integral becomes

$$u_\epsilon(y_\epsilon)^{\frac{2}{2-s_\epsilon}} \leq C \frac{\mu_\epsilon^n}{\mu_\epsilon^{\frac{n-2}{2-s_\epsilon}(2^*(s_\epsilon)-1-s_\epsilon)}} \int_{\frac{\Omega - x_\epsilon}{\mu_\epsilon}} \frac{1}{|y_\epsilon - x_\epsilon - \mu_\epsilon x|^{\frac{2(n-2)}{2-s_\epsilon}}} \hat{v}_\epsilon(x)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{2}{2-s_\epsilon}}\, dx$$

And so we get that for $\epsilon > 0$ small

$$\begin{aligned} \left(\mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon(y_\epsilon)\right)^{\frac{2}{2-s_\epsilon}} \leq & C \int_{\frac{\Omega - x_\epsilon}{\mu_\epsilon} \cap \left\{|y_\epsilon - x_\epsilon - \mu_\epsilon x| \geq \frac{|y_\epsilon - x_\epsilon|}{2}\right\}} \frac{1}{|y_\epsilon - x_\epsilon - \mu_\epsilon x|^{\frac{2(n-2)}{2-s_\epsilon}}} \hat{v}_\epsilon(x)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{2}{2-s_\epsilon}}\, dx \\ & + C \int_{\frac{\Omega - x_\epsilon}{\mu_\epsilon} \cap \left\{|y_\epsilon - x_\epsilon - \mu_\epsilon x| \leq \frac{|y_\epsilon - x_\epsilon|}{2}\right\}} \frac{1}{|y_\epsilon - x_\epsilon - \mu_\epsilon x|^{\frac{2(n-2)}{2-s_\epsilon}}} \hat{v}_\epsilon(x)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{2}{2-s_\epsilon}}\, dx \end{aligned} \tag{4.83}$$

We estimate the above two integrals separately. First we have for $\epsilon > 0$ small

$$\begin{aligned}
&\int_{\frac{\Omega - x_\epsilon}{\mu_\epsilon} \cap \left\{|y_\epsilon - x_\epsilon - \mu_\epsilon x| \geq \frac{|y_\epsilon - x_\epsilon|}{2}\right\}} \frac{1}{|y_\epsilon - x_\epsilon - \mu_\epsilon x|^{\frac{2(n-2)}{2-s_\epsilon}}} \hat{v}_\epsilon(x)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{2}{2-s_\epsilon}} \, dx \\
&\leq \frac{2^{\frac{2(n-2)}{2-s_\epsilon}}}{|y_\epsilon - x_\epsilon|^{\frac{2(n-2)}{2-s_\epsilon}}} \int_{\frac{\Omega - x_\epsilon}{\mu_\epsilon}} \hat{v}_\epsilon(x)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{2}{2-s_\epsilon}} \, dx \\
&\leq \frac{\tilde{C}_\alpha}{|y_\epsilon - x_\epsilon|^{\frac{2(n-2)}{2-s_\epsilon}}} \int_{\frac{\Omega - x_\epsilon}{\mu_\epsilon}} \left(\frac{1}{1+|x|^\alpha}\right)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{2}{2-s_\epsilon}} \, dx
\end{aligned}$$

For α close to $n-2$ we have

$$\int_{\frac{\Omega - x_\epsilon}{\mu_\epsilon}} \left(\frac{1}{1+|x|^\alpha}\right)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{2}{2-s_\epsilon}} \, dx = O(1) \qquad as\ \epsilon \to 0$$

So we obtain that

(4.84)
$$\int_{\frac{\Omega - x_\epsilon}{\mu_\epsilon} \cap \left\{|y_\epsilon - x_\epsilon - \mu_\epsilon x| \geq \frac{|y_\epsilon - x_\epsilon|}{2}\right\}} \frac{\hat{v}_\epsilon^{2^*(s_\epsilon)-1}(x)}{|y_\epsilon - x_\epsilon - \mu_\epsilon x|^{n-2}} \, dx = O\left(\frac{1}{|y_\epsilon - x_\epsilon|^{\frac{2(n-2)}{2-s_\epsilon}}}\right) \qquad as\ \epsilon \to 0$$

On the other hand for $\epsilon > 0$ small

$$\begin{aligned}
&\int_{\frac{\Omega - x_\epsilon}{\mu_\epsilon} \cap \left\{|y_\epsilon - x_\epsilon - \mu_\epsilon x| \leq \frac{|y_\epsilon - x_\epsilon|}{2}\right\}} \frac{1}{|y_\epsilon - x_\epsilon - \mu_\epsilon x|^{\frac{2(n-2)}{2-s_\epsilon}}} \hat{v}_\epsilon(x)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{2}{2-s_\epsilon}} \, dx \\
&\leq C_\alpha \int_{\frac{\Omega - x_\epsilon}{\mu_\epsilon} \cap \left\{|y_\epsilon - x_\epsilon - \mu_\epsilon x| \leq \frac{|y_\epsilon - x_\epsilon|}{2}\right\}} \frac{1}{|y_\epsilon - x_\epsilon - \mu_\epsilon x|^{\frac{2(n-2)}{2-s_\epsilon}}} \frac{1}{|x|^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{2\alpha}{2-s_\epsilon}}} \, dx \\
&\leq C_\alpha \left(\frac{2\mu_\epsilon}{|y_\epsilon - x_\epsilon|}\right)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{2\alpha}{2-s_\epsilon}} \int_{\left\{|y_\epsilon - x_\epsilon - \mu_\epsilon x| \leq \frac{|y_\epsilon - x_\epsilon|}{2}\right\}} \frac{1}{|y_\epsilon - x_\epsilon - \mu_\epsilon x|^{\frac{2(n-2)}{2-s_\epsilon}}} \, dx \\
&\leq C_\alpha \left(\frac{2\mu_\epsilon}{|y_\epsilon - x_\epsilon|}\right)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{2\alpha}{2-s_\epsilon}} \frac{1}{\mu_\epsilon^n} \int_{\left\{|x| \leq \frac{|y_\epsilon - x_\epsilon|}{2}\right\}} \frac{1}{|x|^{\frac{2(n-2)}{2-s_\epsilon}}} \, dx \\
&\leq \tilde{C}_\alpha \left(\frac{\mu_\epsilon}{|y_\epsilon - x_\epsilon|}\right)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{2\alpha}{2-s_\epsilon}} \frac{|y_\epsilon - x_\epsilon|^{\frac{4-ns_\epsilon}{2-s_\epsilon}}}{\mu_\epsilon^n} \\
&\leq \tilde{C}_\alpha \left(\frac{\mu_\epsilon}{|y_\epsilon - x_\epsilon|}\right)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{2\alpha}{2-s_\epsilon} - n} \left(\frac{1}{|y_\epsilon - x_\epsilon|^{n-2}}\right)^{\frac{2}{2-s_\epsilon}}
\end{aligned}$$

Since $\lim_{\alpha\to n-2}\left[(2^*(s_\epsilon)-1-s_\epsilon)\frac{2\alpha}{2-s_\epsilon}-n\right]=\left[2-\frac{n-2}{2-s_\epsilon}s_\epsilon\right]$ for each $\epsilon>0$, so taking α close to $(n-2)$, we obtain for ϵ sufficiently small

$$\int_{\frac{\Omega-x_\epsilon}{\mu_\epsilon}\cap\left\{|y_\epsilon-x_\epsilon-\mu_\epsilon x|\le\frac{|y_\epsilon-x_\epsilon|}{2}\right\}}\frac{\hat{v}_\epsilon^{2^*(s_\epsilon)-1}(x)}{|y_\epsilon-x_\epsilon-\mu_\epsilon x|^{n-2}}\,dx=o(1)\left(\frac{1}{|y_\epsilon-x_\epsilon|^{n-2}}\right)^{\frac{2}{2-s_\epsilon}}\qquad as\ \epsilon\to 0$$

$$as\ \lim_{\epsilon\to 0}\frac{|x_\epsilon-y_\epsilon|}{\mu_\epsilon}=+\infty\qquad as\ \epsilon\to 0 \tag{4.85}$$

Combining (4.83), (4.84) and (4.85) we obtain that

$$\left(\mu_\epsilon^{-\frac{n-2}{2}}u_\epsilon(y_\epsilon)\right)^{\frac{2}{2-s_\epsilon}}\le O\left(\frac{1}{|y_\epsilon-x_\epsilon|^{\frac{2(n-2)}{2-s_\epsilon}}}\right)\qquad as\ \epsilon\to 0$$

And

$$|y_\epsilon-x_\epsilon|^{n-2}\,\mu_\epsilon^{-\frac{n-2}{2}}u_\epsilon(y_\epsilon)\le O\left(1\right)\qquad as\ \epsilon\to 0$$

This proves (4.81) and ends Step 2.2 and then Step 2. $\square$

Step 3: In (4.81) we have obtained that there exists $C>0$ such that for all $\epsilon>0$

$$|x-x_\epsilon|^{n-2}\,\mu_\epsilon^{-\frac{n-2}{2}}u_\epsilon(x)\le C\qquad\text{for all } x\in\Omega$$

By definition (4.14), it then get that for all $\epsilon>0$

$$\left(\mu_\epsilon^2+|x-x_\epsilon|^2\right)^{\frac{n-2}{2}}\mu_\epsilon^{-\frac{n-2}{2}}u_\epsilon(x)\le C\qquad\text{for all } x\in\Omega$$

This completes the proof of Theorem 4.6. $\square$

4.7. Localizing the Singularity: The Interior Blow-up Case

In this section, we prove the following:

Theorem 4.7. *Let Ω be a bounded smooth oriented domain of $\mathbb{R}^n$, $n\ge 3$, such that $0\in\partial\Omega$, and let $a\in C^1(\overline{\Omega})$ be such that the operator $\Delta+a$ is coercive in Ω. Let $(s_\epsilon)_{\epsilon>0}\in(0,2)$ be a sequence such that $\lim_{\epsilon\to 0}s_\epsilon=0$. Suppose that the sequence $(u_\epsilon)_{\epsilon>0}\in H^2_{1,0}(\Omega)$, where for each $\epsilon>0$, u_ϵ satisfies (4.4) and (4.5), is a* blowup sequence, *i.e*

$$u_\epsilon\rightharpoonup 0\qquad weakly\ in\ H^2_{1,0}(\Omega)\qquad as\ \ \epsilon\to 0$$

We let $(\mu_\epsilon)_\epsilon\in(0,+\infty)$ and $(x_\epsilon)_\epsilon\in\Omega$ be such that

$$\mu_\epsilon^{-\frac{n-2}{2}}=u_\epsilon(x_\epsilon)=\max_{x\in\Omega}u_\epsilon(x).$$

We define $x_0:=\lim_{\epsilon\to 0}x_\epsilon$ and we assume that

$$x_0\in\Omega\ is\ an\ interior\ point.$$

Then

$$\lim_{\epsilon\to 0}\frac{s_\epsilon}{\mu_\epsilon^2} = 2^* K(n,0)^{\frac{2^*}{2^*-2}} d_n\ a(x_0) \qquad \textit{for}\ \ n\geq 5$$

$$\lim_{\epsilon\to 0}\frac{s_\epsilon}{\mu_\epsilon^2\log(1/\mu_\epsilon)} = 256\omega_3 K(4,0)^2\ a(x_0) \qquad \textit{for}\ \ n=4$$

$$\lim_{\epsilon\to 0}\frac{s_\epsilon}{\mu_\epsilon^{n-2}} = -nb_n^2 K(n,0)^{n/2} g^a_{x_0}(x_0) \qquad \textit{for}\ n=3\ \textit{or}\ a\equiv 0.$$

where $g^a_{x_0}(x_0)$ *the mass at the point* $x_0\in\Omega$ *for the operator* $\Delta + a$, *where*

$$d_n = \int_{\mathbb{R}^n}\frac{1}{\left(1+\frac{|x|^2}{n(n-2)}\right)^{n-2}}\,dx \quad \textit{for}\ n\geq 5\ ;\ b_n = \int_{\mathbb{R}^n}\frac{1}{\left(1+\frac{|x|^2}{n(n-2)}\right)^{\frac{(n+2)}{2}}}\,dx$$

and ω_3 *is the area of the* 3- *sphere.*

The proof goes through six steps.

Step 1: We first state and prove the celebrated Pohozaev identity.

Lemma 4.7.1 (Pohozaev Identity)**.** *Let* U *be a bounded smooth domain in* $\mathbb{R}^n$, *let* $p_0\in\mathbb{R}^n$ *be a point and let* $u\in C^2(\overline{U})$. *We have*

(4.86)

$$\int_U\left((x-p_0,\nabla u)+\frac{n-2}{2}u\right)\Delta u\ dx = \int_{\partial U}\left((x-p_0,\nu)\frac{|\nabla u|^2}{2} - \left((x-p_0,\nabla u)+\frac{n-2}{2}u\right)\partial_\nu u\right)d\sigma$$

here ν *is the outer normal to the boundary* ∂U.

Proof. Integration by parts gives us

$$\int_U\left((x-p_0,\nabla u)+\frac{n-2}{2}u\right)\Delta u\ dx = -\int_U\left((x-p_0,\nabla u)+\frac{n-2}{2}u\right)\partial_j\partial_j u\ dx$$

$$= \int_U\partial_j\left((x-p_0,\nabla u)+\frac{n-2}{2}u\right)\partial_j u\ dx - \int_{\partial U}\left((x-p_0,\nabla u)+\frac{n-2}{2}u\right)\partial_\nu u\ d\sigma$$

$$= \frac{n}{2}\int_U|\nabla u|^2\ dx + \frac{1}{2}\int_U(x-p_0)^j\partial_j|\nabla u|^2\ dx - \int_{\partial U}\left((x-p_0,\nabla u)+\frac{n-2}{2}u\right)\partial_\nu u\ d\sigma$$

$$= \int_U\partial_j\left((x-p_0)^j\frac{|\nabla u|^2}{2}\right)\ dx - \int_{\partial U}\left((x-p_0,\nabla u)+\frac{n-2}{2}u\right)\partial_\nu u\ d\sigma$$

$$= \int_{\partial U}(x-p_0,\nu)\frac{|\nabla u|^2}{2}\ d\sigma - \int_{\partial U}\left((x-p_0,\nabla u)+\frac{n-2}{2}u\right)\partial_\nu u\ d\sigma$$

$$= \int_{\partial U}\left((x-p_0,\nu)\frac{|\nabla u|^2}{2} - \left((x-p_0,\nabla u)+\frac{n-2}{2}u\right)\partial_\nu u\right)d\sigma$$

□

Step 2: Next using the above Pohozaev Identity we obtain

Lemma 4.7.2. *Let $B_{x_0}(\delta) \subset \Omega$. We have for all $\epsilon > 0$*

$$\int_{B_{x_\epsilon}(\delta)} \left(a + \frac{(x - x_\epsilon, \nabla a)}{2}\right) u_\epsilon^2 \, dx - \frac{s_\epsilon(n-2)}{2(n-s_\epsilon)} \int_{B_{x_\epsilon}(\delta)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \frac{(x, x_\epsilon)}{|x|^2} \, dx =$$

(4.87)

$$\int_{\partial B_{x_\epsilon}(\delta)} (x - x_\epsilon, \nu) \left(\frac{|\nabla u_\epsilon|^2}{2} + \frac{a u_\epsilon^2}{2} - \frac{1}{2^*(s_\epsilon)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \right) d\sigma - \int_{\partial B_{x_\epsilon}(\delta)} \left((x - x_\epsilon, \nabla u_\epsilon) + \frac{n-2}{2} u_\epsilon \right) \partial_\nu u_\epsilon \, d\sigma$$

Proof. One has for $1 \le j \le n$

$$\partial_j \left(\frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \right) = \frac{2(n-s_\epsilon)}{n-2} \frac{u_\epsilon^{2^*(s_\epsilon)-1}}{|x|^{s_\epsilon}} \partial_j u_\epsilon - s_\epsilon \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon+2}} x^j$$

And so

$$\begin{aligned}
(x - x_\epsilon, \nabla u_\epsilon) \frac{u_\epsilon^{2^*(s_\epsilon)-1}}{|x|^{s_\epsilon}} &= \frac{n-2}{2(n-s_\epsilon)} (x - x_\epsilon)^j \partial_j \left(\frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \right) + \frac{s_\epsilon(n-2)}{2(n-s_\epsilon)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon+2}} x^j (x - x_\epsilon)^j \\
&= \frac{n-2}{2(n-s_\epsilon)} (x - x_\epsilon)^j \partial_j \left(\frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \right) + \frac{s_\epsilon(n-2)}{2(n-s_\epsilon)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} - \frac{s_\epsilon(n-2)}{2(n-s_\epsilon)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon+2}} x^j x_\epsilon^j
\end{aligned}$$

Integration by parts gives us

$$\begin{aligned}
\int_{B_{x_\epsilon}(\delta)} (x - x_\epsilon, \nabla u_\epsilon) \frac{u_\epsilon^{2^*(s_\epsilon)-1}}{|x|^{s_\epsilon}} \, dx &= \frac{n-2}{2(n-s_\epsilon)} \int_{B_{x_\epsilon}(\delta)} (x - x_\epsilon)^j \partial_j \left(\frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \right) \, dx \\
&+ \frac{s_\epsilon(n-2)}{2(n-s_\epsilon)} \int_{B_{x_\epsilon}(\delta)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \, dx - \frac{s_\epsilon(n-2)}{2(n-s_\epsilon)} \int_{B_{x_\epsilon}(\delta)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon+2}} (x, x_\epsilon) \, dx \\
&= \frac{-n(n-2)}{2(n-s_\epsilon)} \int_{B_{x_\epsilon}(\delta)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \, dx + \frac{n-2}{2(n-s_\epsilon)} \int_{\partial B_{x_\epsilon}(\delta)} (x - x_\epsilon, \nu) \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \, d\sigma \\
&+ \frac{s_\epsilon(n-2)}{2(n-s_\epsilon)} \int_{B_{x_\epsilon}(\delta)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \, dx - \frac{s_\epsilon(n-2)}{2(n-s_\epsilon)} \int_{B_{x_\epsilon}(\delta)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon+2}} (x, x_\epsilon) \, dx \\
&= -\frac{(n-2)}{2} \int_{B_{x_\epsilon}(\delta)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \, dx - \frac{s_\epsilon(n-2)}{2(n-s_\epsilon)} \int_{B_{x_\epsilon}(\delta)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon+2}} (x, x_\epsilon) \, dx \\
&+ \frac{1}{2^*(s)} \int_{\partial B_{x_\epsilon}(\delta)} (x - x_\epsilon, \nu) \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \, d\sigma
\end{aligned}$$

For $\epsilon > 0$, u_ϵ satisfies the equation

$$\Delta u_\epsilon = \frac{u_\epsilon^{2^*(s_\epsilon)-1}}{|x|^{s_\epsilon}} - a u_\epsilon \qquad \text{in } B_{x_\epsilon}(\delta)$$

Therefore

$$\int_{B_{x_\epsilon}(\delta)} \left((x-x_\epsilon,\nabla u_\epsilon)+\frac{n-2}{2}u_\epsilon\right)\Delta u_\epsilon\, dx = \int_{B_{x_\epsilon}(\delta)} \left((x-x_\epsilon,\nabla u_\epsilon)+\frac{n-2}{2}u_\epsilon\right)\left(\frac{u_\epsilon^{2^*(s_\epsilon)-1}}{|x|^{s_\epsilon}}-au_\epsilon\right) dx$$

$$= \int_{B_{x_\epsilon}(\delta)} \left((x-x_\epsilon,\nabla u_\epsilon)\frac{u_\epsilon^{2^*(s_\epsilon)-1}}{|x|^{s_\epsilon}}+\frac{n-2}{2}\frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}-(x-x_\epsilon,\nabla u_\epsilon)au_\epsilon-\frac{n-2}{2}au_\epsilon^2\right) dx$$

$$= -\frac{(n-2)}{2}\int_{B_{x_\epsilon}(\delta)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\, dx - \frac{s_\epsilon(n-2)}{2(n-s_\epsilon)}\int_{B_{x_\epsilon}(\delta)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon+2}}(x,x_\epsilon)\, dx + \frac{1}{2^*(s)}\int_{\partial B_{x_\epsilon}(\delta)} (x-x_\epsilon,\nu)\frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\, d\sigma$$

$$+\int_{B_{x_\epsilon}(\delta)} \left(\frac{n-2}{2}\frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}-(x-x_\epsilon,\nabla u_\epsilon)au_\epsilon-\frac{n-2}{2}au_\epsilon^2\right) dx$$

$$= -\frac{s_\epsilon(n-2)}{2(n-s_\epsilon)}\int_{B_{x_\epsilon}(\delta)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon+2}}(x,x_\epsilon)\, dx - \int_{B_{x_\epsilon}(\delta)} \left((x-x_\epsilon,\nabla u_\epsilon)au_\epsilon+\frac{n-2}{2}au_\epsilon^2\right)\, dx$$

$$+\frac{1}{2^*(s)}\int_{\partial B_{x_\epsilon}(\delta)} (x-x_\epsilon,\nu)\frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\, d\sigma$$

$$= -\frac{s_\epsilon(n-2)}{2(n-s_\epsilon)}\int_{B_{x_\epsilon}(\delta)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon+2}}(x,x_\epsilon)\, dx + \int_{B_{x_\epsilon}(\delta)} \left(a+\frac{(x-x_\epsilon,\nabla a)}{2}\right)u_\epsilon^2\, dx$$

$$+\frac{1}{2^*(s)}\int_{\partial B_{x_\epsilon}(\delta)} (x-x_\epsilon,\nu)\frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\, d\sigma - \int_{\partial B_{x_\epsilon}(\delta)} (x-x_\epsilon,\nu)\frac{au_\epsilon^2}{2}\, d\sigma$$

Using the Pohozaev identity (4.86) we then have the lemma. □

Since $x_0\in\Omega$, let $\delta>0$ be such that $B_{x_0}(3\delta)\subset\Omega$. Note that then $\lim\limits_{\epsilon\to 0}|x_\epsilon|^{s_\epsilon}=1$, and it follows from (4.17) that $\lim\limits_{\epsilon\to 0}\mu_\epsilon^{s_\epsilon}=1$.

We will estimate each of the terms in the above Pohozaev identity and calculate the limit as $\epsilon\to$ and $\delta\to 0$. It will depend on the dimension n. Let $G^a:\overline{\Omega}\times\overline{\Omega}\setminus\{(x,x):x\in\overline{\Omega}\}\longrightarrow\mathbb{R}$ be the Green's function of the coercive operator $\Delta+a$ in Ω with Dirichlet boundary conditions. For existence and the properties of G^a see Ghoussoub-Robert [**9**] (Theorem B.1) and Robert [**17**]. For a fixed point x, let $G^a_x(y)=G^a(x,y)$ for $y\in\overline{\Omega}\setminus\{x\}$.

Step 3: We prove the following convergence outside x_0:

Proposition 4.7.1.

$$\mu_\epsilon^{-\frac{n-2}{2}}u_\epsilon\longrightarrow b_nG^a_{x_0}\qquad in\ C^1_{loc}(\overline{\Omega}\setminus\{x_0\})\qquad as\ \epsilon\to 0$$

where $b_n=\displaystyle\int_{\mathbb{R}^n}\frac{1}{\left(1+\frac{|x|^2}{n(n-2)}\right)^{\frac{n+2}{2}}}\, dx$

PROOF. **Step 3.1:** We fix $y_0 \in \Omega$ such that $y_0 \neq x_0$. We claim that

$$\lim_{\epsilon \to 0} \mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon(y_0) \longrightarrow b_n G^a_{x_0}(y_0)$$

where $b_n = \int\limits_{\mathbb{R}^n} \frac{1}{\left(1 + \frac{|x|^2}{n(n-2)}\right)^{\frac{n+2}{2}}} \, dx.$

We prove the claim. We choose $\delta' \in (0, \delta)$ such that $|x_0 - y_0| \geq 3\delta'$ and $|x_0| \geq 3\delta'$. Fom *Green's representation formula* we have

$$\mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon(y_0) = \mu_\epsilon^{-\frac{n-2}{2}} \int\limits_{\Omega} G^a(x, y_0) \frac{u_\epsilon^{2^*(s_\epsilon)-1}(x)}{|x|^{s_\epsilon}} \, dx \qquad \text{for all } \epsilon > 0$$

$$= \mu_\epsilon^{-\frac{n-2}{2}} \int\limits_{B_{x_\epsilon}(\delta')} G^a(x, y_0) \frac{u_\epsilon^{2^*(s_\epsilon)-1}(x)}{|x|^{s_\epsilon}} \, dx + \mu_\epsilon^{-\frac{n-2}{2}} \int\limits_{\Omega \setminus B_{x_\epsilon}(\delta')} G^a(x, y_0) \frac{u_\epsilon^{2^*(s_\epsilon)-1}(x)}{|x|^{s_\epsilon}} \, dx$$

In theorem 4.6 we have obtained that there exists a constant $C > 0$ such that for all $x \in \Omega$,
$u_\epsilon(x) \leq \frac{C\mu_\epsilon^{\frac{n-2}{2}}}{|x - x_\epsilon|^{n-2}}$ for all $\epsilon > 0$. Then using the estimates on the Green's function G^a we obtain as $\epsilon \to 0$

$$\int\limits_{\Omega \setminus B_{x_\epsilon}(\delta')} G^a(x, y_0) \frac{u_\epsilon^{2^*(s_\epsilon)-1}(x)}{|x|^{s_\epsilon}} \, dx = O(\mu_\epsilon^{\frac{n+2(1-s_\epsilon)}{2}}) \int\limits_{\Omega \setminus B_{x_\epsilon}(\delta')} \frac{1}{|x - y_0|^{n-2}} \frac{1}{|x|^{s_\epsilon}} \, dx$$

$$= O(\mu_\epsilon^{\frac{n+2(1-s_\epsilon)}{2}}) \left(\int\limits_{\Omega} \frac{1}{|x - y_0|^{\frac{2(n-2)}{2-s_\epsilon}}} \, dx \right)^{\frac{2-s_\epsilon}{2}} \left(\int\limits_{\Omega} \frac{1}{|x|^2} \, dx \right)^{\frac{s_\epsilon}{2}}$$

$$= O(\mu_\epsilon^{\frac{n+2(1-s_\epsilon)}{2}})$$

So we have for $\epsilon > 0$ small

$$\mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon(y_0) = \mu_\epsilon^{-\frac{n-2}{2}} \int\limits_{B_{x_\epsilon}(\delta')} G^a(x, y_0) \frac{u_\epsilon^{2^*(s_\epsilon)-1}(x)}{|x|^{s_\epsilon}} \, dx + O(\mu_\epsilon^{2-s_\epsilon})$$

Recall our definition of v_ϵ in theorem 4.4. With a change of variable we then obtain from Theorem 4.4

$$\mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon(y_0) = \mu_\epsilon^{-\frac{n-2}{2}} \int\limits_{B_{x_\epsilon}(\delta')} G^a(x, y_0) \frac{u_\epsilon^{2^*(s_\epsilon)-1}(x)}{|x|^{s_\epsilon}} \, dx + O(\mu_\epsilon^{2-s_\epsilon})$$

$$= \left(\frac{|x_\epsilon|^{s_\epsilon}}{\mu_\epsilon^{s_\epsilon}} \right)^{\frac{n-2}{2}} \int\limits_{B_0(\delta' k_\epsilon^{-1})} G^a(x_\epsilon + k_\epsilon x, y_0) \frac{v_\epsilon^{2^*(s_\epsilon)-1}(x)}{\left| \frac{x_\epsilon}{|x_\epsilon|} + \frac{k_\epsilon}{|x_\epsilon|} x \right|^{s_\epsilon}} \, dx + O(\mu_\epsilon^{2-s_\epsilon})$$

We have that $\lim_{\epsilon \to 0} \frac{|x_\epsilon|^{s_\epsilon}}{\mu_\epsilon^{s_\epsilon}} = 1$ and from theorem 4.6, it follows that there exists a constant $C > 0$ such that as $\epsilon \to 0$

$$v_\epsilon(x) \leq C \left(\frac{1}{1 + \left(\frac{k_\epsilon}{\mu_\epsilon}\right)^2 |x|^2} \right)^{\frac{n-2}{2}} \leq C \left(\frac{1}{1 + \frac{|x|^2}{n(n-2)}} \right)^{\frac{n-2}{2}}$$

and so

$$v_\epsilon^{2*(s_\epsilon)-1}(x) \leq C\left(\frac{1}{1+\frac{|x|^2}{n(n-2)}}\right)^{\frac{n+2}{2}}\left(1+\frac{|x|^2}{n(n-2)}\right)^{s_\epsilon} \leq C\left(\frac{1}{1+|x|^2}\right)^{\frac{n+2}{2}}\left(1+\frac{|x|^2}{n(n-2)}\right)^{1/2}$$

The integral $\displaystyle\int_{\mathbb{R}^n} \frac{\left(1+\frac{|x|^2}{n(n-2)}\right)^{1/2}}{\left(1+\frac{|x|^2}{n(n-2)}\right)^{\frac{n+2}{2}}}\, dx$ is finite. Also we have that $\frac{1}{2} \leq \left|\frac{x_\epsilon}{|x_\epsilon|}+\frac{k_\epsilon}{|x_\epsilon|}x\right| \leq \frac{3}{2}$ and $|G^a(x_\epsilon + k_\epsilon x, y_0)| \leq \frac{1}{\delta'^{n-2}}$ for all $x \in B_0(\delta' k_\epsilon^{-1})$. Therefore by Lebesgue dominated convergence theorem and Theorem 4.4 it follows that

$$\lim_{\epsilon\to 0} \mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon(y_0) = G^a(x_0,y_0)\int_{\mathbb{R}^n} v^{2^*-1}dx = G^a(x_0,y_0)\int_{\mathbb{R}^n} \frac{1}{\left(1+\frac{|x|^2}{n(n-2)}\right)^{\frac{n+2}{2}}}\, dx.$$

This proves the claim and ends Step 3.1.

Step 3.2: Let $\Omega' \subset\subset \Omega'' \subset\subset \overline{\Omega}\setminus\{x_0\}$ be a compactly contained open sets. From (4.4) it follows that for $\epsilon > 0$ the functions $\mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon$ satisfies the equation

$$\begin{aligned} \Delta(\mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon) + a(x)(\mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon) &= \mu_\epsilon^{2-s_\epsilon}\frac{(\mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon)^{2^*(s_\epsilon)-1}}{|x|^{s_\epsilon}} \qquad \text{in } \mathscr{D}'(\Omega'') \\ \mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon &= 0 \qquad \text{on } \Omega''\cap\partial\Omega \end{aligned}$$

In Theorem 4.6 we have obtained that there exists a constant $C > 0$ such that for all $x \in \Omega$, $\mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon(x) \leq \frac{C}{|x-x_\epsilon|^{n-2}}$ for all $\epsilon > 0$, and so $\|u_\epsilon\|_{L^\infty(\Omega'')} = O(1)$ as $\epsilon \to 0$ and $\mu_\epsilon^{2-s_\epsilon}\frac{(\mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon)^{2^*(s_\epsilon)-1}}{|x|^{s_\epsilon}} \in L^p(\Omega'')$ uniformly for some $p > n$ and $\epsilon > 0$ small. Then from standard elliptic estimates (see for instance [**14**]) it follows that $\|\mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon\|_{C^{1,\alpha}(\Omega')} = O(1)$ as $\epsilon \to 0$. Hence the sequence $(\mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon)_{\epsilon>0}$ is precompact in $C^1(\overline{\Omega'})$. In our previous step we have show that $\lim_{\epsilon\to 0} \mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon(y) \longrightarrow b_n G^a_{x_0}(y)$ for every $y \in \Omega'$, therefore it follows that

$$\lim_{\epsilon\to 0} \mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon \longrightarrow b_n G^a_{x_0} \qquad \text{in } C^1(\Omega')$$

This completes the proof of Proposition 4.7.1. $\square$

Step 4: Next we show that

$$\lim_{\epsilon\to 0}\int_{B_{x_\epsilon}(\delta)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\frac{(x,x_\epsilon)}{|x|^2}\, dx = \left(\frac{1}{K(n,0)}\right)^{\frac{2^*}{2^*-2}} \tag{4.88}$$

Proof. Recall our definition of v_ϵ in Theorem 4.4. With a change of variable we have

$$\int_{B_{x_\epsilon}(\delta)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\frac{(x,x_\epsilon)}{|x|^2}\, dx = \left(\frac{|x_\epsilon|^{s_\epsilon}}{\mu_\epsilon^{s_\epsilon}}\right)^{\frac{n-2}{2}} \int_{B_0(\delta/k_\epsilon)} \frac{(x_\epsilon + k_\epsilon x, x_\epsilon)}{|x_\epsilon + k_\epsilon x|^2}\frac{v_\epsilon(x)^{2^*(s_\epsilon)}}{\left|\frac{x_\epsilon}{|x_\epsilon|}+\frac{k_\epsilon}{|x_\epsilon|}x\right|^{s_\epsilon}}\, dx$$

We have obtained earlier in theorem (4.4) that $\lim_{\epsilon\to 0}\frac{\mu_\epsilon^{s_\epsilon}}{|x_\epsilon|^{s_\epsilon}}=1$, $\lim_{\epsilon\to 0}\frac{r_\epsilon}{k_\epsilon}=+\infty$ and $\lim_{\epsilon\to 0}\frac{k_\epsilon}{|x_\epsilon|}=0$. Also we have for all $x\in B_0(\delta/k_\epsilon)$

$$1=\left|\frac{x_\epsilon}{|x_\epsilon|}\right|\leq\left|\frac{x_\epsilon}{|x_\epsilon|}+\frac{k_\epsilon}{|x_\epsilon|}x\right|+\frac{k_\epsilon}{|x_\epsilon|}|x|\leq\left|\frac{x_\epsilon}{|x_\epsilon|}+\frac{k_\epsilon}{|x_\epsilon|}x\right|+\frac{\delta}{|x_\epsilon|}\leq\left|\frac{x_\epsilon}{|x_\epsilon|}+\frac{k_\epsilon}{|x_\epsilon|}x\right|+\frac{1}{3}$$

So for all $x\in B_0(\delta/k_\epsilon)$

$$\left|\frac{x_\epsilon}{|x_\epsilon|}+\frac{k_\epsilon}{|x_\epsilon|}x\right|\geq\frac{2}{3}$$

Then passing to limits, and using Theorems 4.4 and 4.6 we obtain by Lebesgue dominated convergence theorem

$$\lim_{\epsilon\to 0}\int_{B_{x_\epsilon}(\delta)}\frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\frac{(x,x_\epsilon)}{|x|^2}\,dx=\int_{\mathbb{R}^n}v^{2^*}\,dx=\left(\frac{1}{K(n,0)}\right)^{\frac{2^*}{2^*-2}}$$

This proves (4.88) and ends Step 4. □

Step 5: We prove Theorem 4.7 for $n\geq 4$.

Using the Pohozaev identity we have obtained in (4.87) that

$$\int_{B_{x_\epsilon}(\delta)}\left(a+\frac{(x-x_\epsilon,\nabla a)}{2}\right)u_\epsilon^2\,dx-\frac{s_\epsilon(n-2)}{2(n-s_\epsilon)}\int_{B_{x_\epsilon}(\delta)}\frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\frac{(x,x_\epsilon)}{|x|^2}\,dx=$$
$$\int_{\partial B_{x_\epsilon}(\delta)}(x-x_\epsilon,\nu)\left(\frac{|\nabla u_\epsilon|^2}{2}+\frac{au_\epsilon^2}{2}-\frac{1}{2^*(s_\epsilon)}\frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\right)d\sigma-\int_{\partial B_{x_\epsilon}(\delta)}\left((x-x_\epsilon,\nabla u_\epsilon)+\frac{n-2}{2}u_\epsilon\right)\partial_\nu u_\epsilon\,d\sigma$$

Step 5.1: we assume here that $n\geq 5$. We have for $\epsilon>0$

$$\mu_\epsilon^{-2}\int_{B_{x_\epsilon}(\delta)}\left(a+\frac{(x-x_\epsilon,\nabla a)}{2}\right)u_\epsilon^2\,dx-\mu_\epsilon^{-2}\frac{s_\epsilon(n-2)}{2(n-s_\epsilon)}\int_{B_{x_\epsilon}(\delta)}\frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\frac{(x,x_\epsilon)}{|x|^2}\,dx$$
$$=\mu_\epsilon^{n-4}\int_{\partial B_{x_\epsilon}(\delta)}(x-x_\epsilon,\nu)\left(\frac{|\nabla(\mu_\epsilon^{-\frac{n-2}{2}}u_\epsilon)|^2}{2}+\frac{a}{2}(\mu_\epsilon^{-\frac{n-2}{2}}u_\epsilon)^2-\frac{\mu_\epsilon^{2-s_\epsilon}}{2^*(s_\epsilon)}\frac{(\mu_\epsilon^{-\frac{n-2}{2}}u_\epsilon)^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\right)d\sigma$$
$$-\mu_\epsilon^{n-4}\int_{\partial B_{x_\epsilon}(\delta)}\left((x-x_\epsilon,\nabla(\mu_\epsilon^{-\frac{n-2}{2}}u_\epsilon))+\frac{n-2}{2}(\mu_\epsilon^{-\frac{n-2}{2}}u_\epsilon)\right)\partial_\nu(\mu_\epsilon^{-\frac{n-2}{2}}u_\epsilon)\,d\sigma \tag{4.89}$$

First we calculate the right hand side of the above equality. Recall our definition of v_ϵ in theorem 4.4. With a change of variable we obtain

$$\mu_\epsilon^{-2}\int_{B_{x_\epsilon}(\delta)}\left(a+\frac{(x-x_\epsilon,\nabla a)}{2}\right)u_\epsilon^2\,dx=\left(\frac{k_\epsilon}{\mu_\epsilon}\right)^n\int_{B_0(\delta k_\epsilon^{-1})}\left(a(x_\epsilon+k_\epsilon x)+\frac{(k_\epsilon x,\nabla a(x_\epsilon+k_\epsilon x))}{2}\right)v_\epsilon^2\,dx$$

We have that $\lim_{\epsilon\to 0}\frac{k_\epsilon}{\mu_\epsilon} = 1$ and from Theorem 4.6, it follows that there exists a constant $C > 0$ such that as $\epsilon \to 0$

$$v_\epsilon(x) \leq C\left(\frac{1}{1+\left(\frac{k_\epsilon}{\mu_\epsilon}\right)^2|x|^2}\right)^{\frac{n-2}{2}} \leq C\left(\frac{1}{1+\frac{|x|^2}{n(n-2)}}\right)^{\frac{n-2}{2}}$$

and so $v_\epsilon^2(x) \leq \frac{C}{\left(1+\frac{|x|^2}{n(n-2)}\right)^{n-2}}$. We have that for $n \geq 5$, the integral $\int\limits_{\mathbb{R}^n} \frac{1}{\left(1+\frac{|x|^2}{n(n-2)}\right)^{n-2}}\, dx$ is finite. We let

$$d_n = \int\limits_{\mathbb{R}^n} \frac{1}{\left(1+\frac{|x|^2}{n(n-2)}\right)^{n-2}}\, dx \qquad \text{for } n \geq 5$$

Therefore

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0}\left[\mu_\epsilon^{-2}\int\limits_{B_{x_\epsilon}(\delta)}\left(a+\frac{(x-x_\epsilon,\nabla a)}{2}\right)u_\epsilon^2\, dx\right] = d_n\ a(x_0) \qquad \text{for } n \geq 5$$

Passing to the limits as $\epsilon \to 0$ in (4.89) we then obtain using proposition 4.7.1 and (4.88)

$$d_n\ a(x_0) - \lim_{\epsilon\to 0}\frac{s_\epsilon}{\mu_\epsilon^2}\frac{1}{2^*}\left(\frac{1}{K(n,0)}\right)^{\frac{2^*}{2^*-2}} = 0$$

and so

$$\lim_{\epsilon\to 0}\frac{s_\epsilon}{\mu_\epsilon^2} = 2^* K(n,0)^{\frac{2^*}{2^*-2}} d_n\ a(x_0)$$

This proves Theorem 4.7 when $n \geq 5$ and ends Step 5.1.

Step 5.2: We now deal with the case $n = 4$. We have for $\epsilon > 0$

$$\frac{\mu_\epsilon^{-2}}{\log(1/k_\epsilon)}\int\limits_{B_{x_\epsilon}(\delta)}\left(a+\frac{(x-x_\epsilon,\nabla a)}{2}\right)u_\epsilon^2\, dx - \frac{\mu_\epsilon^{-2}}{\log(1/k_\epsilon)}\frac{s_\epsilon}{4-s_\epsilon}\int\limits_{B_{x_\epsilon}(\delta)}\frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\frac{(x,x_\epsilon)}{|x|^2}\, dx$$

$$= \frac{1}{\log(1/k_\epsilon)}\int\limits_{\partial B_{x_\epsilon}(\delta)}(x-x_\epsilon,\nu)\left(\frac{|\nabla(\mu_\epsilon^{-1}u_\epsilon)|^2}{2}+\frac{a}{2}(\mu_\epsilon^{-1}u_\epsilon)^2-\frac{\mu_\epsilon^{2-s_\epsilon}}{2^*(s_\epsilon)}\frac{(\mu_\epsilon^{-1}u_\epsilon)^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\right)d\sigma$$

$$-\frac{1}{\log(1/k_\epsilon)}\int\limits_{\partial B_{x_\epsilon}(\delta)}\left((x-x_\epsilon,\nabla(\mu_\epsilon^{-1}u_\epsilon))+\mu_\epsilon^{-1}u_\epsilon\right)\partial_\nu(\mu_\epsilon^{-1}u_\epsilon)\, d\sigma \tag{4.90}$$

With a change of variable we obtain

$$\frac{\mu_\epsilon^{-2}}{\log(1/k_\epsilon)}\int\limits_{B_{x_\epsilon}(\delta)}\left(a+\frac{(x-x_\epsilon,\nabla a)}{2}\right)u_\epsilon^2\, dx$$

$$= \frac{1}{\log(1/k_\epsilon)}\left(\frac{k_\epsilon}{\mu_\epsilon}\right)^4\int\limits_{B_0(\delta k_\epsilon^{-1})}\left(a(x_\epsilon+k_\epsilon x)+\frac{(k_\epsilon x,\nabla a(x_\epsilon+k_\epsilon x))}{2}\right)v_\epsilon^2\, dx$$

We have that $\lim_{\epsilon\to 0}\frac{k_\epsilon}{\mu_\epsilon} = 1$ and from theorem 4.6, it follows that there exists a constant $C>0$ such that as $\epsilon\to 0$

$$v_\epsilon(x) \le C\frac{1}{1+\left(\frac{k_\epsilon}{\mu_\epsilon}\right)^2|x|^2} \le C\frac{1}{1+\frac{|x|^2}{n(n-2)}}$$

and so $v_\epsilon^2(x) \le \frac{C}{\left(1+\frac{|x|^2}{n(n-2)}\right)^2}$. We have

$$\lim_{\epsilon\to 0}\left[\frac{1}{\log(1/k_\epsilon)}\int_{B_0(\delta k_\epsilon^{-1})}\frac{1}{\left(1+\frac{|x|^2}{8}\right)^2}\,dx\right] = 64\omega_3$$

Hence

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0}\left[\frac{\mu_\epsilon^{-2}}{\log(1/k_\epsilon)}\int_{B_{x_\epsilon}(\delta)}\left(a+\frac{(x-x_\epsilon,\nabla a)}{2}\right)u_\epsilon^2\,dx\right] = 64\omega_3 a(x_0) \qquad \text{for } n=4.$$

Passing to the limits as $\epsilon\to 0$ in (4.90) we then obtain using Proposition 4.7.1 and (4.88)

$$64\omega_3 a(x_0) - \lim_{\epsilon\to 0}\frac{s_\epsilon}{4\mu_\epsilon^2\log(1/k_\epsilon)}\left(\frac{1}{K(4,0)}\right)^2 = 0$$

and so

$$\lim_{\epsilon\to 0}\frac{s_\epsilon}{\mu_\epsilon^2\log(1/k_\epsilon)} = 256\omega_3 K(4,0)^2\; a(x_0)$$

This proves Theorem 4.7 for $n=4$, and therefore ends Step 5.2 and Step 5.

Step 6: We now deal with the case of dimension $n=3$. Recall from the introduction that we write the Green's function G^a as

$$G_x^a(y) = \frac{1}{4\pi|x-y|} + g_x^a(y) \text{ for all } x,y\in\Omega,\ x\neq y$$

and $g_x^a \in C^2(\overline{\Omega}\setminus\{x\})\cap C^{0,\theta}(\Omega)$ for some $0<\theta<1$, and g^a is called the regular part of the Green's function G^a. In particular, when $n=3$ or $a\equiv 0$, $m_x(\Omega,a) := g_x^a(x)$ is defined for all $x\in\Omega$ and is called the mass of the operator $\Delta + a$. Note that for any $x\in\Omega$, g_x^a satifies the equation

$$\begin{aligned} \Delta g_x^a + aG_x^a &= 0 && \textit{in } \Omega\setminus\{x\} \\ g_x^a(y) &= \frac{-1}{\omega_2|x-y|} && \textit{on } \partial\Omega \end{aligned}$$

For any $x\in\Omega$, we claim that

$$\lim_{r\to 0}\sup_{y\in\partial B_x(r)}|y-x||\nabla g_x^a(y)| = 0 \tag{4.91}$$

The proof goes as in Hebey-Robert [**12**]. We omit it here.

We now exploit the Pohozaev identity. Using the Pohozaev identity we have obtained in (4.87) that

$$\int_{B_{x_\epsilon}(\delta)} \left(a + \frac{(x-x_\epsilon, \nabla a)}{2}\right) u_\epsilon^2 \, dx - \frac{s_\epsilon(n-2)}{2(n-s_\epsilon)} \int_{B_{x_\epsilon}(\delta)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \frac{(x, x_\epsilon)}{|x|^2} \, dx =$$
$$\int_{\partial B_{x_\epsilon}(\delta)} (x-x_\epsilon, \nu) \left(\frac{|\nabla u_\epsilon|^2}{2} + \frac{a u_\epsilon^2}{2} - \frac{1}{2^*(s_\epsilon)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\right) d\sigma - \int_{\partial B_{x_\epsilon}(\delta)} \left((x-x_\epsilon, \nabla u_\epsilon) + \frac{n-2}{2} u_\epsilon\right) \partial_\nu u_\epsilon \, d\sigma$$

Multiplying both the sides by μ_ϵ^{-1} we obtain

$$\int_{B_{x_\epsilon}(\delta)} \left(a + \frac{(x-x_\epsilon, \nabla a)}{2}\right) (\mu_\epsilon^{-1/2} u_\epsilon)^2 \, dx - \frac{s_\epsilon}{2\mu_\epsilon(3-s_\epsilon)} \int_{B_{x_\epsilon}(\delta)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \frac{(x, x_\epsilon)}{|x|^2} \, dx =$$
$$\int_{\partial B_{x_\epsilon}(\delta)} (x-x_\epsilon, \nu) \left(\frac{|\nabla (\mu_\epsilon^{-1/2} u_\epsilon)|^2}{2} + a\frac{(\mu_\epsilon^{-1/2} u_\epsilon)^2}{2} - \frac{\mu_\epsilon^{2-s_\epsilon}}{2^*(s_\epsilon)} \frac{(\mu_\epsilon^{-1/2} u_\epsilon)^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\right) d\sigma$$
$$(4.92)$$
$$- \int_{\partial B_{x_\epsilon}(\delta)} \left((x-x_\epsilon, \nabla (\mu_\epsilon^{-1/2} u_\epsilon)) + \frac{n-2}{2} (\mu_\epsilon^{-1/2} u_\epsilon)\right) \partial_\nu (\mu_\epsilon^{-1/2} u_\epsilon) \, d\sigma$$

It follows from Proposition 4.7.1 that

$$\lim_{\epsilon \to 0} \int_{\partial B_{x_\epsilon}(\delta)} (x-x_\epsilon, \nu) \left(\frac{|\nabla (\mu_\epsilon^{-1/2} u_\epsilon)|^2}{2} + a\frac{(\mu_\epsilon^{-1/2} u_\epsilon)^2}{2} - \frac{\mu_\epsilon^{2-s_\epsilon}}{2^*(s_\epsilon)} \frac{(\mu_\epsilon^{-1/2} u_\epsilon)^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\right) d\sigma$$
$$- \lim_{\epsilon \to 0} \int_{\partial B_{x_\epsilon}(\delta)} \left((x-x_\epsilon, \nabla (\mu_\epsilon^{-1/2} u_\epsilon)) + \frac{n-2}{2} (\mu_\epsilon^{-1/2} u_\epsilon)\right) \partial_\nu (\mu_\epsilon^{-1/2} u_\epsilon) \, d\sigma$$
$$= b_3^2 \int_{\partial B_{x_0}(\delta)} \left((x-x_0, \nu) \left(\frac{|\nabla G^a_{x_0}|^2}{2} + \frac{a}{2} (G^a_{x_0})^2\right) - \left((x-x_0, \nabla G^a_{x_0}) + \frac{n-2}{2} G^a_{x_0}\right) \partial_\nu G^a_{x_0}\right) \, d\sigma$$
$$= b_3^2 \int_{\partial B_{x_0}(\delta)} \delta \frac{|\nabla G^a_{x_0}|^2}{2} + \frac{\delta a}{2} (G^a_{x_0})^2 - \frac{(x-x_0, \nabla G^a_{x_0})^2}{\delta} - \frac{n-2}{2} \frac{(x-x_0, \nabla G^a_{x_0})}{\delta} G^a_{x_0} \, d\sigma$$

One has for $n = 3$

$$\begin{aligned}
G_{x_0}(x) &= \frac{1}{(n-2)\omega_{n-1}} \frac{1}{|x-x_0|^{n-2}} + g_{x_0}(x),\\
\partial_j G_{x_0}(x) &= -\frac{1}{\omega_{n-1}} \frac{1}{|x-x_0|^{n-1}} \frac{(x-x_0)^j}{|x-x_0|} + \partial_j g_{x_0}(x) \qquad \text{for } 1 \le j \le n,\\
|\nabla G_{x_0}(x)|^2 &= \frac{1}{\omega_{n-1}^2} \frac{1}{|x-x_0|^{2n-2}} - \frac{1}{\omega_{n-1}} \frac{2}{|x-x_0|^{n-1}} \frac{(x-x_0, \nabla g_{x_0})}{|x-x_0|} + |\nabla g_{x_0}(x)|^2,\\
(x-x_0, \nabla G_{x_0}(x)) &= -\frac{1}{\omega_{n-1}} \frac{1}{|x-x_0|^{n-2}} + (x-x_0, \nabla g_{x_0}(x)),\\
(x-x_0, \nabla G_{x_0}(x))^2 &= \frac{1}{\omega_{n-1}^2} \frac{1}{|x-x_0|^{2n-4}} - \frac{1}{\omega_{n-1}} \frac{2}{|x-x_0|^{n-2}} (x-x_0, \nabla g_{x_0}(x)) + (x-x_0, \nabla g_{x_0}(x))^2,\\
(x-x_0, \nabla G_{x_0}(x)) G_{x_0}(x) &= -\frac{1}{(n-2)\omega_{n-1}^2} \frac{1}{|x-x_0|^{2n-4}} - \frac{1}{\omega_{n-1}} \frac{1}{|x-x_0|^{n-2}} g_{x_0}(x)\\
&\quad + (x-x_0, \nabla g_{x_0}(x)) \frac{1}{(n-2)\omega_{n-1}} \frac{1}{|x-x_0|^{n-2}} + (x-x_0, \nabla g_{x_0}(x)) g_{x_0}(x)
\end{aligned}$$

Then we have

$$\begin{aligned}
&b_3^2 \int\limits_{\partial B_{x_0}(\delta)} \delta \frac{|\nabla G_{x_0}^a|^2}{2} + \frac{\delta a}{2} (G_{x_0}^a)^2 - \frac{(x-x_0, \nabla G_{x_0}^a)^2}{\delta} - \frac{n-2}{2} \frac{(x-x_0, \nabla G_{x_0}^a)}{\delta} G_{x_0}^a \, d\sigma\\
&= b_3^2 \int\limits_{\partial B_{x_0}(\delta)} \frac{1}{2\omega_2^2 \delta^3} - \frac{1}{\omega_2 \delta^2} (x-x_0, \nabla g_{x_0}^a(x)) + \delta \frac{|\nabla g_{x_0}^a(x)|^2}{2} + \frac{a}{\omega_2^2 \delta} + a g_{x_0}^a + \frac{\delta (g_{x_0}^a)^2}{2} \, d\sigma\\
&+ b_3 \int\limits_{\partial B_{x_0}(\delta)} -\frac{1}{\omega_2^2 \delta^3} + \frac{2}{\omega_2 \delta^2} (x-x_0, \nabla g_{x_0}^a(x)) - \frac{(x-x_0, \nabla g_{x_0}^a(x))^2}{\delta} \, d\sigma\\
&+ b_3^2 \int\limits_{\partial B_{x_0}(\delta)} \frac{1}{2\omega_2^2 \delta^3} + \frac{1}{2\omega_2 \delta^2} g_{x_0}^a(x) - \frac{1}{2\omega_2 \delta^2} (x-x_0, \nabla g_{x_0}^a(x)) - \frac{1}{2\delta} (x-x_0, \nabla g_{x_0}^a(x)) g_{x_0}^a(x) \, d\sigma\\
&= b_3^2 \int\limits_{\partial B_{x_0}(\delta)} \frac{1}{2\omega_2 \delta^2} g_{x_0}^a(x) \, d\sigma + b_3 \int\limits_{\partial B_{x_0}(\delta)} \frac{1}{2\omega_2 \delta^2} (x-x_0, \nabla g_{x_0}^a(x)) \, d\sigma + b_3 \int\limits_{\partial B_{x_0}(\delta)} \frac{a}{\omega_2^2 \delta} + a g_{x_0}^a + \frac{\delta (g_{x_0}^a)^2}{2} \, d\sigma\\
&+ b_3^2 \int\limits_{\partial B_{x_0}(\delta)} \left[\delta \frac{|\nabla g_{x_0}^a(x)|^2}{2} - \frac{(x-x_0, \nabla g_{x_0}^a(x))^2}{\delta} - \frac{1}{2\delta} (x-x_0, \nabla g_{x_0}^a(x)) g_{x_0}^a(x) \right] d\sigma
\end{aligned}$$

Using (4.91) it then follows that

$$\begin{aligned}
&\lim_{\delta \to 0} \left[b_3^2 \int\limits_{\partial B_{x_0}(\delta)} \delta \frac{|\nabla G_{x_0}^a|^2}{2} + \frac{\delta a}{2} (G_{x_0}^a)^2 - \frac{(x-x_0, \nabla G_{x_0}^a)^2}{\delta} - \frac{n-2}{2} \frac{(x-x_0, \nabla G_{x_0}^a)}{\delta} G_{x_0}^a \, d\sigma \right]\\
&= \frac{b_3^2}{2} g_{x_0}^a(x_0)
\end{aligned}$$

In theorem 4.6 we have obtained that there exists a constant $C > 0$ such that for all $x \in \Omega$, $\mu_\epsilon^{-1/2} u_\epsilon(x) \leq \frac{C}{|x-x_\epsilon|}$ for all $\epsilon > 0$. So we obtain

$$\int\limits_{B_{x_\epsilon}(\delta)} \left| a + \frac{(x - x_\epsilon, \nabla a)}{2} \right| (\mu_\epsilon^{-1/2} u_\epsilon)^2 \, dx \leq C \int\limits_{B_{x_\epsilon}(\delta)} (\mu_\epsilon^{-1/2} u_\epsilon(x))^2 \, dx$$

$$\leq C \int\limits_{B_0(\delta)} \frac{1}{|x|^2} \, dx \leq C\delta$$

And hence we have

$$\lim_{\delta \to 0} \lim_{\epsilon \to 0} \int\limits_{B_{x_\epsilon}(\delta)} \left(a + \frac{(x - x_\epsilon, \nabla a)}{2} \right) (\mu_\epsilon^{-1/2} u_\epsilon)^2 \, dx = 0$$

Plugging all these together in (4.92) and using (4.88) we then have

$$\lim_{\epsilon \to 0} \frac{s_\epsilon}{\mu_\epsilon} = -3b_3^2 m_{x_0}(\Omega, a) K(3,0)^{3/2}$$

This proves Theorem 4.7 in the case $n = 3$.

4.8. Localizing the Singularity: The Boundary Blow-up Case

This section is devoted to the proof of the following result:

Theorem 4.8. *Let Ω be a bounded smooth oriented domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$, and let $a \in C^1(\overline{\Omega})$ be such that the operator $\Delta + a$ is coercive in Ω. Let $(s_\epsilon)_{\epsilon > 0} \in (0, 2)$ be a sequence such that $\lim_{\epsilon \to 0} s_\epsilon = 0$. Suppose that the sequence $(u_\epsilon)_{\epsilon > 0} \in H_{1,0}^2(\Omega)$, where for each $\epsilon > 0$, u_ϵ satisfies (4.4) and (4.5), is a* blowup sequence*, i.e*

$$u_\epsilon \rightharpoonup 0 \qquad \text{weakly in } H_{1,0}^2(\Omega) \qquad \text{as } \epsilon \to 0$$

We let $(\mu_\epsilon)_\epsilon \in (0, +\infty)$ and $(x_\epsilon)_\epsilon \in \Omega$ be such that

$$\mu_\epsilon^{-\frac{n-2}{2}} = u_\epsilon(x_\epsilon) = \max_{x \in \Omega} u_\epsilon(x).$$

We define $x_0 := \lim_{\epsilon \to 0} x_\epsilon$ and we assume that

$$x_0 \in \partial\Omega \text{ is a boundary point.}$$

Then

(1) *If $n = 3$ or $a \equiv 0$, then as $\epsilon \to 0$*

$$\lim_{\epsilon \to 0} \frac{s_\epsilon d(x_\epsilon, \partial\Omega)^{n-2}}{\mu_\epsilon^{n-2}} = \frac{n^{n-1}(n-2)^{n-1} K(n,0)^{n/2}}{2^{n-2}}.$$

(2) *If $n = 4$. Then as $\epsilon \to 0$*

$$\frac{s_\epsilon}{4} \left(K(4,0)^{-2} + o(1) \right) - \left(\frac{\mu_\epsilon}{d(x_\epsilon, \partial\Omega)} \right)^2 (32\omega_3 + o(1)) = \mu_\epsilon^2 \log \left(\frac{d(x_\epsilon, \partial\Omega)}{k_\epsilon} \right) [64\omega_3 a(x_0) + o(1)]$$

(3) *If* $n \geq 5$. *Then as* $\epsilon \to 0$

$$\frac{s_\epsilon(n-2)}{2n}\left(K(n,0)^{-n/2}+o(1)\right)-\left(\frac{\mu_\epsilon}{d(x_\epsilon,\partial\Omega)}\right)^{n-2}\left(\frac{n^{n-2}(n-2)^n}{2^{n-1}}+o(1)\right)=\mu_\epsilon^2\left[d_n a(x_0)+o(1)\right]$$

where

$$d_n=\int_{\mathbb{R}^n}\frac{1}{\left(1+\frac{|x|^2}{n(n-2)}\right)^{n-2}}\,dx \quad \text{for } n\geq 5 \qquad \text{and } d_4=64\omega_3$$

4.8.1. Convergence to Singular Harmonic Functions. Let $G^a:\overline{\Omega}\times\overline{\Omega}\backslash\{(x,x): x\in\overline{\Omega}\}\longrightarrow\mathbb{R}$ be the Green's function of the coercive operator $\Delta+a$ in Ω with Dirichlet boundary conditions. For existence and the properties of G^a see Ghoussoub-Robert [**9**] (Theorem B.1) and Robert [**17**]. For a fixed point x, we let $G^a_x(y)=G^a(x,y)$ for $y\in\overline{\Omega}\backslash\{x\}$. One has the following result for the asymptotic analysis of the Green's function G^a, the proof of which is in *Proposition 5* of [**17**] and *Proposition 12* of [**7**].

Theorem 4.9 ([**7**, **17**])**.** *Let* $(x_\epsilon)_{\epsilon>0}\in\Omega$ *and let* $(r_\epsilon)_{\epsilon>0}\in(0,+\infty)$ *be such that* $\lim_{\epsilon\to 0} r_\epsilon=0$.

(1) *If*

$$\lim_{\epsilon\to 0}\frac{d(x_\epsilon,\partial\Omega)}{r_\epsilon}=+\infty$$

Then for all $x,y\in\mathbb{R}^n$, $x\neq y$, *we have that*

$$\lim_{\epsilon\to 0} r_\epsilon^{n-2}G^a(x_\epsilon+r_\epsilon x, x_\epsilon+r_\epsilon y)=\frac{1}{(n-2)\omega_{n-1}|x-y|^{n-2}}$$

where ω_{n-1} *is the area of the* $(n-1)$*- sphere. Moreover for a fixed* $x\in\mathbb{R}^n$, *this convergence holds uniformly in* $C^2_{loc}(\mathbb{R}^n\backslash\{x\})$.

(2) *If*

$$\lim_{\epsilon\to 0}\frac{d(x_\epsilon,\partial\Omega)}{r_\epsilon}=\rho\in[0,+\infty)$$

Then $\lim_{\epsilon\to 0} x_\epsilon=x_0\in\partial\Omega$. *Let* $\mathcal{T}$ *be a parametrisation of the boundary* $\partial\Omega$ *as in* (4.18) *around the point* $p=x_0$. *We write* $\mathcal{T}^{-1}(x_\epsilon)=((x_\epsilon)_1,x'_\epsilon)$. *Then for all* $x,y\in\mathbb{R}^n\cap\{x_1\leq 0\}$, $x\neq y$, *we have that*

$$\begin{aligned}&\lim_{\epsilon\to 0} r_\epsilon^{n-2}G^a\left(\mathcal{T}((0,x'_\epsilon)+r_\epsilon x),\mathcal{T}((0,x'_\epsilon)+r_\epsilon y)\right)\\&=\frac{1}{(n-2)\omega_{n-1}|x-y|^{n-2}}-\frac{1}{(n-2)\omega_{n-1}|\pi(x)-y|^{n-2}}\end{aligned}$$

where $\pi:\mathbb{R}^n\to\mathbb{R}^n$ *defined by* $\pi((x_1,x'))\mapsto(-x_1,x')$ *is the reflection across the plane* $\{x: x_1=0\}$. *Moreover for a fixed* $x\in\overline{\mathbb{R}^n_-}$, *this convergence holds uniformly in* $C^2_{loc}(\overline{\mathbb{R}^n_-}\backslash\{x\})$.

Next we show that the pointwise behaviour of the blow up sequence $(u_\epsilon)_{\epsilon>0}$ is well approximated by bubbles. Note that the following proposition holds with $x_0\in\overline{\Omega}$, in the interior or on the boundary.

Proposition 4.8.1. *We set for all $\epsilon > 0$*

$$U_\epsilon(x) = \left(\frac{k_\epsilon}{k_\epsilon^2 + \frac{|x-x_\epsilon|^2}{n(n-2)}} \right)^{\frac{n-2}{2}}$$

Suppose that the sequence $(u_\epsilon)_{\epsilon>0} \in H_{1,0}^2(\Omega)$, where for each $\epsilon > 0$, u_ϵ satisfies (4.4) and (4.5), is a blowup sequence. *We let $x_0 := \lim_{\epsilon\to 0} x_\epsilon$. Let $(y_\epsilon)_{\epsilon>0}$ be a sequence of points in $\overline{\Omega}$. We have*

(1) *If $\lim_{\epsilon\to 0} y_\epsilon = y_0 \neq x_0$, then*

$$\lim_{\epsilon\to 0} \mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon(y_\epsilon) = b_n G^a(x_0, y_0)$$

where $b_n := \int\limits_{\mathbb{R}^n} \frac{1}{\left(1+\frac{|x|^2}{n(n-2)}\right)^{\frac{n+2}{2}}}\, dx$.

(2) *If $\lim_{\epsilon\to 0} y_\epsilon = x_0$ and $\lim\limits_{\epsilon\to 0} d(x_\epsilon, \partial\Omega) > 0$, then*

$$u(y_\epsilon) = (1 + o(1)) U_\epsilon(y_\epsilon) \quad \text{as } \epsilon \to 0$$

(3) *If $\lim_{\epsilon\to 0} y_\epsilon = x_0$ and $\lim\limits_{\epsilon\to 0} d(x_\epsilon, \partial\Omega) = 0$, then*

$$u(y_\epsilon) = (1 + o(1)) \left(U_\epsilon(y_\epsilon) - \tilde{U}_\epsilon(y_\epsilon) \right) \quad \text{as } \epsilon \to 0$$

where for $\epsilon > 0$

$$\tilde{U}_\epsilon(x) = \left(\frac{k_\epsilon}{k_\epsilon^2 + \frac{|x-\pi_{\mathcal{T}}(x_\epsilon)|^2}{n(n-2)}} \right)^{\frac{n-2}{2}}$$

with $\pi_{\mathcal{T}} = \mathcal{T} \circ \pi \circ \mathcal{T}^{-1}$, $\mathcal{T}$ is a parametrisation of the boundary $\partial\Omega$ as in (4.18) around the point $p = x_0 \in \partial\Omega$ where $\lim\limits_{\epsilon\to 0} x_\epsilon = x_0 \in \partial\Omega$. And, $\pi : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\pi((x_1, x')) \mapsto (-x_1, x')$ is the reflection across the plane $\{x : x_1 = 0\}$.

PROOF. It follows from *Green's representation formula* that

$$u_\epsilon(y_\epsilon) = \int\limits_\Omega G^a(x, y_\epsilon) \frac{u_\epsilon^{2^*(s_\epsilon)-1}(x)}{|x|^{s_\epsilon}}\, dx \qquad \text{for all } \epsilon > 0$$

Case (1), that is $\lim_{\epsilon\to 0} y_\epsilon = y_0 \neq 0$, is dealt in Proposition 4.7.1. We now deal with the case

$$\lim_{\epsilon\to 0} |x_\epsilon - y_\epsilon| = 0.$$

Step 1: We claim that

$$\lim_{R\to+\infty} \lim_{\epsilon\to 0} \frac{\int\limits_{\Omega\setminus B_{x_\epsilon}(Rk_\epsilon)} G^a(x, y_\epsilon) \frac{u_\epsilon^{2^*(s_\epsilon)-1}(x)}{|x|^{s_\epsilon}}\, dx}{U_\epsilon(y_\epsilon)} = 0 \tag{4.93}$$

PROOF. It follows from the estimates on the Green's function G^a that there exists a constant $C > 0$ such that for all $\epsilon > 0$

$$\int_{\Omega \setminus B_{x_\epsilon}(Rk_\epsilon)} G^a(x, y_\epsilon) \frac{u_\epsilon^{2^*(s_\epsilon)-1}(x)}{|x|^{s_\epsilon}} \, dx \le C \int_{\Omega \setminus B_{x_\epsilon}(R\mu_\epsilon)} \frac{1}{|x - y_\epsilon|^{n-2}} \frac{u_\epsilon^{2^*(s_\epsilon)-1}(x)}{|x|^{s_\epsilon}} \, dx$$
$$\le C \int_{\Omega \setminus B_{x_\epsilon}(R\mu_\epsilon)} \left(\frac{u_\epsilon(x)}{|x|} \right)^{s_\epsilon} \frac{1}{|x - y_\epsilon|^{n-2}} \, u_\epsilon(x)^{2^*(s_\epsilon)-1-s_\epsilon} \, dx$$

Using Hölder inequality and then the Hardy inequality (4.8) we then obtain for all $\epsilon > 0$

$$\int_{\Omega \setminus B_{x_\epsilon}(R\mu_\epsilon)} G^a(x, y_\epsilon) \frac{u_\epsilon^{2^*(s_\epsilon)-1}(x)}{|x|^{s_\epsilon}} \, dx$$
$$\le C \left(\int_\Omega \frac{|u_\epsilon(x)|^2}{|x|^2} \, dx \right)^{s_\epsilon/2} \left(\int_{\Omega \setminus B_{x_\epsilon}(R\mu_\epsilon)} \left(\frac{1}{|x - y_\epsilon|^{n-2}} \right)^{\frac{2}{2-s_\epsilon}} u_\epsilon(x)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{2}{2-s_\epsilon}} \, dx \right)^{\frac{2-s_\epsilon}{2}}$$
$$\le C \left(\left(\frac{2}{n-2} \right)^2 \int_\Omega |\nabla u_\epsilon|^2 \, dx \right)^{s_\epsilon/2} \left(\int_{\Omega \setminus B_{x_\epsilon}(R\mu_\epsilon)} \left(\frac{1}{|x - y_\epsilon|^{n-2}} \right)^{\frac{2}{2-s_\epsilon}} u_\epsilon(x)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{2}{2-s_\epsilon}} \, dx \right)^{\frac{2-s_\epsilon}{2}}$$

The sequence $(u_\epsilon)_{\epsilon>0}$ is bounded in $H^2_{1,0}(\Omega)$, so it follows that there exists a constant $C > 0$ such that for $\epsilon > 0$ small

$$\int_{\Omega \setminus B_{x_\epsilon}(R\mu_\epsilon)} G^a(x, y_\epsilon) \frac{u_\epsilon^{2^*(s_\epsilon)-1}(x)}{|x|^{s_\epsilon}} \, dx \le C \left(\int_{\Omega \setminus B_{x_\epsilon}(R\mu_\epsilon)} \left(\frac{1}{|x - y_\epsilon|^{n-2}} \right)^{\frac{2}{2-s_\epsilon}} u_\epsilon(x)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{2}{2-s_\epsilon}} \, dx \right)^{\frac{2-s_\epsilon}{2}}$$

By the strong pointwise bound on u_ϵ we then have for $\epsilon > 0$ small

$$\int_{\Omega \setminus B_{x_\epsilon}(R\mu_\epsilon)} G^a(x, y_\epsilon) \frac{u_\epsilon^{2^*(s_\epsilon)-1}(x)}{|x|^{s_\epsilon}} \, dx \le$$
$$C \left(\int_{\Omega \setminus B_{x_\epsilon}(R\mu_\epsilon)} \left(\frac{1}{|x - y_\epsilon|^{n-2}} \right)^{\frac{2}{2-s_\epsilon}} \left(\frac{\mu_\epsilon}{\mu_\epsilon^2 + |x - x_\epsilon|^2} \right)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{n-2}{2-s_\epsilon}} dx \right)^{\frac{2-s_\epsilon}{2}}$$

Let

$$D_\epsilon = \left\{ x \in \mathbb{R}^n : |x - y_\epsilon| \ge \frac{1}{2} \sqrt{\mu_\epsilon^2 + |x_\epsilon - y_\epsilon|^2} \right\}$$

We split the above integral into two terms

$$\int_{\Omega\backslash B_{x_\epsilon}(R\mu_\epsilon)} \left(\frac{1}{|x-y_\epsilon|^{n-2}}\right)^{\frac{2}{2-s_\epsilon}} \left(\frac{\mu_\epsilon}{\mu_\epsilon^2+|x-x_\epsilon|^2}\right)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{n-2}{2-s_\epsilon}} dx =$$

$$\int_{D_\epsilon\cap(\Omega\backslash B_{x_\epsilon}(R\mu_\epsilon))} \left(\frac{1}{|x-y_\epsilon|^{n-2}}\right)^{\frac{2}{2-s_\epsilon}} \left(\frac{\mu_\epsilon}{\mu_\epsilon^2+|x-x_\epsilon|^2}\right)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{n-2}{2-s_\epsilon}} dx$$

$$+ \int_{(\mathbb{R}^n\backslash D_\epsilon)\cap(\Omega\backslash B_{x_\epsilon}(R\mu_\epsilon))} \left(\frac{1}{|x-y_\epsilon|^{n-2}}\right)^{\frac{2}{2-s_\epsilon}} \left(\frac{\mu_\epsilon}{\mu_\epsilon^2+|x-x_\epsilon|^2}\right)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{n-2}{2-s_\epsilon}} dx$$

We have for some constant $C>0$

$$\int_{D_\epsilon\cap(\Omega\backslash B_{x_\epsilon}(R\mu_\epsilon))} \left(\frac{1}{|x-y_\epsilon|^{n-2}}\right)^{\frac{2}{2-s_\epsilon}} \left(\frac{\mu_\epsilon}{\mu_\epsilon^2+|x-x_\epsilon|^2}\right)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{n-2}{2-s_\epsilon}} dx$$

$$\leq \frac{C}{(\mu_\epsilon^2+|x_\epsilon-y_\epsilon|^2)^{\frac{n-2}{2-s_\epsilon}}} \int_{\mathbb{R}^n\backslash B_{x_\epsilon}(R\mu_\epsilon)} \left(\frac{\mu_\epsilon}{\mu_\epsilon^2+|x-x_\epsilon|^2}\right)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{n-2}{2-s_\epsilon}} dx$$

$$\leq C\left(\frac{\mu_\epsilon}{(\mu_\epsilon^2+|x_\epsilon-y_\epsilon|^2)}\right)^{\frac{n-2}{2-s_\epsilon}} \int_{\mathbb{R}^n\backslash B_0(R)} \left(\frac{1}{1+|x|^2}\right)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{n-2}{2-s_\epsilon}} dx$$

On the other hand there exists $C'>0$ such that for $x\notin D_\epsilon$ we have for $\epsilon>0$

$$|y_\epsilon-x_\epsilon|^2+\mu_\epsilon^2 \leq C'\left(|x-x_\epsilon|^2+\mu_\epsilon^2\right)$$

Consequently for some $C > 0$

$$\int_{(\mathbb{R}^n\setminus D_\epsilon)\cap(\Omega\setminus B_{x_\epsilon}(R\mu_\epsilon))} \left(\frac{1}{|x-y_\epsilon|^{n-2}}\right)^{\frac{2}{2-s_\epsilon}} \left(\frac{\mu_\epsilon}{\mu_\epsilon^2+|x-x_\epsilon|^2}\right)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{n-2}{2-s_\epsilon}} dx$$

$$\leq C\left(\frac{\mu_\epsilon}{\mu_\epsilon^2+|y_\epsilon-x_\epsilon|^2}\right)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{n-2}{2-s_\epsilon}} \int_{(\mathbb{R}^n\setminus D_\epsilon)\cap(\Omega\setminus B_{x_\epsilon}(R\mu_\epsilon))} \left(\frac{1}{|x-y_\epsilon|^{n-2}}\right)^{\frac{2}{2-s_\epsilon}} dx$$

$$\leq C\left(\frac{\mu_\epsilon}{\mu_\epsilon^2+|y_\epsilon-x_\epsilon|^2}\right)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{n-2}{2-s_\epsilon}} \int_{|x-y_\epsilon|\leq\frac{1}{2}\sqrt{\mu_\epsilon^2+|x_\epsilon-y_\epsilon|^2}} \left(\frac{1}{|x-y_\epsilon|^{n-2}}\right)^{\frac{2}{2-s_\epsilon}} dx$$

$$\leq C\left(\frac{\mu_\epsilon}{\mu_\epsilon^2+|x_\epsilon-y_\epsilon|^2}\right)^{\frac{n-2}{2-s_\epsilon}} \left(\frac{\mu_\epsilon}{\mu_\epsilon^2+|y_\epsilon-x_\epsilon|^2}\right)^{(2^*(s_\epsilon)-2-s_\epsilon)\frac{n-2}{2-s_\epsilon}} (\mu_\epsilon^2+|x_\epsilon-y_\epsilon|^2)^{\frac{4-ns_\epsilon}{2(2-s_\epsilon)}}$$

$$\leq C\left(\frac{\mu_\epsilon}{\mu_\epsilon^2+|x_\epsilon-y_\epsilon|^2}\right)^{\frac{n-2}{2-s_\epsilon}} \left(\frac{\mu_\epsilon}{\mu_\epsilon^2+|y_\epsilon-x_\epsilon|^2}\right)^{\frac{4-ns_\epsilon}{2-s_\epsilon}} (\mu_\epsilon^2+|x_\epsilon-y_\epsilon|^2)^{\frac{4-ns_\epsilon}{2(2-s_\epsilon)}}$$

$$\leq C\left(\frac{\mu_\epsilon}{\mu_\epsilon^2+|x_\epsilon-y_\epsilon|^2}\right)^{\frac{n-2}{2-s_\epsilon}} \left(\frac{\mu_\epsilon^2}{\mu_\epsilon^2+|y_\epsilon-x_\epsilon|^2}\right)^{\frac{4-ns_\epsilon}{2(2-s_\epsilon)}}$$

$$\leq C\left(\frac{\mu_\epsilon}{\mu_\epsilon^2+|x_\epsilon-y_\epsilon|^2}\right)^{\frac{n-2}{2-s_\epsilon}} \left(\frac{1}{1+\frac{|y_\epsilon-x_\epsilon|^2}{\mu_\epsilon^2}}\right)^{\frac{4-ns_\epsilon}{2(2-s_\epsilon)}}$$

(4.94)

$$= o\left(\left(\frac{\mu_\epsilon}{\mu_\epsilon^2+|x_\epsilon-y_\epsilon|^2}\right)^{\frac{n-2}{2-s_\epsilon}}\right) \qquad \text{If } \mu_\epsilon = o\left(|x_\epsilon-y_\epsilon|\right) \text{ as } \epsilon\to 0$$

In case $|x_\epsilon - y_\epsilon| = O(\mu_\epsilon)$ as $\epsilon \to 0$, then for R large , $(\mathbb{R}^n \setminus D_\epsilon) \cap (\Omega \setminus B_{x_\epsilon}(R\mu_\epsilon) = \emptyset$ for all epsilon $\epsilon > 0$. So (4.94) always holds Combining, we then have for $\epsilon > 0$ small

$$\int_{\Omega\setminus B_{x_\epsilon}(R\mu_\epsilon)} G^a(x,y_\epsilon)\frac{u_\epsilon^{2^*(s_\epsilon)-1}(x)}{|x|^{s_\epsilon}}\,dx \leq$$

$$C\left(\frac{\mu_\epsilon}{(\mu_\epsilon^2+|x_\epsilon-y_\epsilon|^2)}\right)^{\frac{n-2}{2}} \left(\int_{\mathbb{R}^n\setminus B_0(R)} \left(\frac{1}{1+|x|^2}\right)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{n-2}{2-s_\epsilon}} dx + o(1)\right)^{\frac{2-s_\epsilon}{2}}$$

$$\leq C\ U_\epsilon(y_\epsilon)\left(\ \epsilon_R + o(1)\right)^{\frac{2-s_\epsilon}{2}}$$

where $\lim\limits_{R\to+\infty} \epsilon_R = 0$. Passing to limits as $\epsilon \to 0$ and $R \to +\infty$, we obtain (4.93). This ends Step 1. □

We have then for $\epsilon > 0$ small and $R > 0$ large

$$u_\epsilon(y_\epsilon) = \int_{B_{x_\epsilon}(Rk_\epsilon)} G^a(x,y_\epsilon)\frac{u_\epsilon^{2^*(s_\epsilon)-1}(x)}{|x|^{s_\epsilon}}\,dx + (o(1)+\epsilon_R)\,U_\epsilon(y_\epsilon)$$

with a change of variable this becomes

$$u_\epsilon(y_\epsilon) = \left(\frac{k_\epsilon}{\mu_\epsilon}\right)^{\frac{n-2}{2}} U_\epsilon(y_\epsilon) \int\limits_{B_0(R)} \left(k_\epsilon^2 + \frac{|y_\epsilon - x_\epsilon|^2}{n(n-2)}\right)^{\frac{n-2}{2}} G^a(y_\epsilon, x_\epsilon + k_\epsilon x) \frac{v_\epsilon^{2^*(s_\epsilon)-1}(x)}{\left|\frac{x_\epsilon}{|x_\epsilon|} + \frac{k_\epsilon}{|x_\epsilon|}x\right|^{s_\epsilon}}\, dx$$
$$+ (o(1) + \epsilon_R)\, U_\epsilon(y_\epsilon)$$

Step 2: We assume that

$$|y_\epsilon - x_\epsilon| = O(k_\epsilon) \text{ as } \epsilon \to 0. \tag{4.95}$$

Let $\theta_\epsilon = \frac{y_\epsilon - x_\epsilon}{k_\epsilon}$, for $\epsilon > 0$ and let $\lim\limits_{\epsilon\to} \theta_\epsilon = \theta_0$. Let K be a compact subset of $\mathbb{R}^n \setminus \{\theta_0\}$. From theorem(4.9) it then follows that as $\epsilon \to 0$.

$$k_\epsilon^{n-2}\, G^a(y_\epsilon, x_\epsilon + k_\epsilon x) = \left(\frac{1}{(n-2)\omega_{n-1}} + o(1)\right) \frac{1}{|x - \theta_0|^{n-2}}$$

uniformly on K. Using the upper bound on G^a, Lebesgue's dominated convergence theorem and $|x_\epsilon|/k_\epsilon \geq d(x_\epsilon, \partial\Omega)/k_\epsilon \to +\infty$ as $\epsilon \to 0$, we have as $\epsilon \to 0$ and $R \to +\infty$

$$u_\epsilon(y_\epsilon) = \left(\frac{k_\epsilon}{\mu_\epsilon}\right)^{\frac{n-2}{2}} U_\epsilon(y_\epsilon) \int\limits_{B_0(R)} \left(k_\epsilon^2 + \frac{|y_\epsilon - x_\epsilon|^2}{n(n-2)}\right)^{\frac{n-2}{2}} G^a(y_\epsilon, x_\epsilon + k_\epsilon x) \frac{v_\epsilon^{2^*(s_\epsilon)-1}(x)}{\left|\frac{x_\epsilon}{|x_\epsilon|} + \frac{k_\epsilon}{|x_\epsilon|}x\right|^{s_\epsilon}}\, dx$$
$$+ (o(1) + \epsilon_R)\, U_\epsilon(y_\epsilon)$$
$$= \left(\frac{1}{(n-2)\omega_{n-1}} \int\limits_{B_0(R)} \left(1 + \frac{|\theta_0|^2}{n(n-2)}\right)^{\frac{n-2}{2}} \frac{1}{|x - \theta_0|^{n-2}}\, v^{2^*-1}(x)\, dx + o(1) + \epsilon_R\right) U_\epsilon(y_\epsilon)$$
$$= \left(\frac{1}{(n-2)\omega_{n-1}} \int\limits_{B_0(R)} \left(1 + \frac{|\theta_0|^2}{n(n-2)}\right)^{\frac{n-2}{2}} \frac{1}{|x - \theta_0|^{n-2}}\, \Delta v(x)\, dx + o(1) + \epsilon_R\right) U_\epsilon(y_\epsilon)$$
$$= (1 + o(1) + \epsilon_R)\, U_\epsilon(y_\epsilon) \qquad \text{by using Green's formula for} v.$$

We remark that in case $\lim\limits_{\epsilon\to 0} d(x_\epsilon, \partial\Omega) = 0$, $\tilde{U}_\epsilon$ are well defined and one has $\frac{\tilde{U}_\epsilon(y_\epsilon)}{U_\epsilon(y_\epsilon)} = o(1)$ as $\epsilon \to 0$ if $|y_\epsilon - x_\epsilon| = O(k_\epsilon)$. This proves Proposition 4.8.1 when (4.95) holds and ends Step 2.

Step 3: We assume that

$$\lim_{\epsilon\to 0} \frac{|y_\epsilon - x_\epsilon|}{k_\epsilon} = +\infty \tag{4.96}$$

Let

$$r_\epsilon = |y_\epsilon - x_\epsilon|$$

Then $r_\epsilon = o(1)$ as $\epsilon \to 0$. For $x \in B_0(R)$ we define for $\epsilon > 0$

$$A_{R,\epsilon} = \left(k_\epsilon^2 + \frac{|y_\epsilon - x_\epsilon|^2}{n(n-2)}\right)^{\frac{n-2}{2}} G^a(y_\epsilon, x_\epsilon + k_\epsilon x)$$

Step 3.1: We assume that $\lim\limits_{\epsilon\to 0} \frac{d(x_\epsilon, \partial\Omega)}{r_\epsilon} = +\infty$.

Let $\theta_\epsilon = \frac{y_\epsilon - x_\epsilon}{r_\epsilon}$, for $\epsilon > 0$ and let $\lim_{\epsilon \to} \theta_\epsilon = \theta_0$. Then $|\theta_0| = 1$. We can write as $\epsilon \to 0$

$$A_{R,\epsilon} = \left(\frac{1}{n(n-2)} + o(1) \right)^{\frac{n-2}{2}} r_\epsilon^{n-2} G^a(x_\epsilon + r_\epsilon \frac{k_\epsilon x}{r_\epsilon}, x_\epsilon + r_\epsilon \theta_\epsilon)$$

Then from Theorem 4.9 we have that

$$\lim_{\epsilon \to 0} A_{R,\epsilon}(x) = \frac{1}{(n(n-2))^{\frac{n-2}{2}}} \frac{1}{(n-2)\omega_{n-1}}$$

uniformly for all x in any fixed compact subset of $\mathbb{R}^n$. This ends Step 3.1.

Step 3.2: We assume that $\lim_{\epsilon \to 0} \frac{d(x_\epsilon, \partial\Omega)}{r_\epsilon} = \rho \in [0, +\infty)$.

In this case the functions $\tilde{U}_\epsilon$ are well defined. Let $x_0 \in \partial\Omega$ be such that $\lim_{\epsilon \to 0} x_\epsilon \to x_0$. Let $\mathcal{T}$ be a parametrisation of the boundary $\partial\Omega$ as in (4.18) around the point $p = x_0$. We write $\mathcal{T}^{-1}(x_\epsilon) = ((x_\epsilon)_1, x'_\epsilon)$ and $\mathcal{T}^{-1}(y_\epsilon) = ((y_\epsilon)_1, y'_\epsilon)$. For $\epsilon > 0$, let

$$X_\epsilon = \left(\frac{(x_\epsilon)_1}{r_\epsilon}, 0 \right) \qquad \textit{and} \qquad Y_\epsilon = \left(\frac{(y_\epsilon)_1}{r_\epsilon}, \frac{y'_\epsilon - x'_\epsilon}{r_\epsilon} \right)$$

Then we get using Theorem 4.9 that for uniformly for all x in any fixed compact subset of R^n

$$\begin{aligned} A_{R,\epsilon} &= \left(\frac{1}{n(n-2)} + o(1) \right)^{\frac{n-2}{2}} r_\epsilon^{n-2} G^a(x_\epsilon + k_\epsilon x_\epsilon, y_\epsilon) \\ &= \left(\frac{1}{n(n-2)} + o(1) \right)^{\frac{n-2}{2}} r_\epsilon^{n-2} G^a(\mathcal{T}((0, x'_\epsilon) + r_\epsilon X_\epsilon), \mathcal{T}((0, x'_\epsilon) + r_\epsilon Y_\epsilon)) \\ &= \left(\frac{1}{n(n-2)} + o(1) \right)^{\frac{n-2}{2}} \frac{1}{(n-2)\omega_{n-1}} \left(\frac{1}{|Y_\epsilon - X_\epsilon|^{n-2}} - \frac{1}{|Y_\epsilon - \pi(X_\epsilon)|^{n-2}} \right) + o(1) \\ &= \left(\frac{1}{n(n-2)} + o(1) \right)^{\frac{n-2}{2}} \frac{1}{(n-2)\omega_{n-1}} \left(1 - \frac{|Y_\epsilon - X_\epsilon|^{n-2}}{|Y_\epsilon - \pi(X_\epsilon)|^{n-2}} \right) + o(1) \end{aligned}$$

since $D_0\mathcal{T}$ is an isometry. Independently we have

$$\begin{aligned} \frac{\tilde{U}_\epsilon(y_\epsilon)}{U_\epsilon(y_\epsilon)} &= \left(\frac{n(n-2)k_\epsilon^2 + |y_\epsilon - \pi_{\mathcal{T}}(x_\epsilon)|^2}{n(n-2)k_\epsilon^2 + |y_\epsilon - x_\epsilon|^2} \right)^{\frac{n-2}{2}} \\ &= (1 + o(1)) \left(\frac{|\mathcal{T}((y_\epsilon)_1, y'_\epsilon) - \mathcal{T}(-(x_\epsilon)_1, x'_\epsilon)|^2}{|\mathcal{T}((y_\epsilon)_1, y'_\epsilon) - \mathcal{T}((x_\epsilon)_1, x'_\epsilon)|^2} \right)^{\frac{n-2}{2}} \\ &= (1 + o(1)) \left(\frac{|((y_\epsilon)_1, y'_\epsilon) - (-(x_\epsilon)_1, x'_\epsilon)|^2}{|((y_\epsilon)_1, y'_\epsilon) - ((x_\epsilon)_1, x'_\epsilon)|^2} \right)^{\frac{n-2}{2}} \\ &= (1 + o(1)) \left(\frac{|Y_\epsilon - \pi(X_\epsilon)|^2}{|Y_\epsilon - X_\epsilon|^2} \right)^{\frac{n-2}{2}} \end{aligned}$$

So it follows as $\epsilon \to 0$

$$A_{R,\epsilon} = \frac{1}{(n(n-2))^{\frac{n-2}{2}}} \frac{1}{(n-2)\omega_{n-1}} \left(1 - \frac{\tilde{U}_\epsilon(y_\epsilon)}{U_\epsilon(y_\epsilon)} \right) + o(1)$$

This ends Step 3.2.

Since for all $R > 0$

$$\int_{B_0(R)} v^{2^*-1}\, dx = \int_{B_0(R)} \Delta v\, dx = -\int_{\partial B_0(R)} \partial_\nu v\, d\sigma = \frac{1}{n}\frac{\omega_{n-1}R^n}{(1+\frac{R^2}{n(n-2)})^{n/2}}$$

we obtain Proposition 4.8.1 in all the cases. □

Using Proposition 4.8.1, we derive the following when the sequence of blowup points converge to a point on the boundary

Proposition 4.8.2. *Let $(u_\epsilon)_{\epsilon>0} \in H^2_{1,0}(\Omega)$ be such that for each $\epsilon > 0$, u_ϵ satisfies* (4.4) *and* (4.5)*. We assume that $u_\epsilon \rightharpoonup 0$ weakly in $H^2_{1,0}(\Omega)$ as $\epsilon \to 0$. We let $x_0 := \lim_{\epsilon\to 0} x_\epsilon$. Let $r_\epsilon = d(x_\epsilon, \partial\Omega)$. We assume that*

$$\lim_{\epsilon\to 0} r_\epsilon = 0.$$

Therefore, $\lim_{\epsilon\to 0} x_\epsilon = x_0 \in \partial\Omega$. Let $\mathcal{T}$ be a parametrisation of the boundary $\partial\Omega$ as in (4.18) *around the point $p = x_0$. We write $\mathcal{T}^{-1}(x_\epsilon) = ((x_\epsilon)_1, x'_\epsilon)$. For $\epsilon > 0$, let*

$$\tilde{v}_\epsilon(x) := \frac{r_\epsilon^{n-2}}{\mu_\epsilon^{\frac{n-2}{2}}} u_\epsilon \circ \mathcal{T}((0, x'_\epsilon) + r_\epsilon x) \qquad \text{for } x \in \frac{U - (0, x'_\epsilon)}{r_\epsilon} \cap \{x_1 \le 0\}$$

Then

$$\lim_{\epsilon\to 0} \tilde{v}_\epsilon(x) = (n(n-2))^{\frac{n-2}{2}} \left(\frac{1}{|x-\theta_0|^{n-2}} - \frac{1}{|x-\pi(\theta_0)|^{n-2}} \right) \text{ in } C^1_{loc}(\overline{\mathbb{R}^n_-} \setminus \{\theta_0\})$$

where

$$\theta_0 = \lim_{\epsilon\to 0} \theta_\epsilon, \qquad \theta_\epsilon = \left(\frac{(x_\epsilon)_1}{r_\epsilon}, 0 \right) \in \mathbb{R}^n_-$$

and $\pi : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\pi((x_1, x')) \mapsto (-x_1, x')$ is the reflection across the plane $\{x : x_1 = 0\}$.

Proof. Since $D_0\mathcal{T} = \mathbb{I}_{\mathbb{R}^n}$ we have

$$d(x_\epsilon, \partial\Omega) = (1 + o(1))\, |(x_\epsilon)_1|$$

Let θ_ϵ be a sequence of points in $\mathbb{R}^n_-$ defined by

$$\theta_\epsilon = \left(\frac{(x_\epsilon)_1}{r_\epsilon}, 0 \right) \qquad \text{for } \epsilon > 0$$

Then it follows that

$$\theta_0 = \lim_{\epsilon\to 0} \theta_\epsilon = (-1, 0) \in \mathbb{R}^n_- \qquad \text{and } \pi(\theta_0) = (1, 0) \in \mathbb{R}^n_+$$

Let $R > 0$. $\tilde{v}_\epsilon$ is defined in $B_0(R) \cap \{x_1 \le 0\}$ for $\epsilon > 0$ small. It follows from the strong upper bounds obtained in Theorem 4.6 that there exists a constant $C > 0$ such that for $\epsilon > 0$ small we have

$$0 \le \tilde{v}_\epsilon(x) \le C \left(\frac{r_\epsilon^2}{|\mathcal{T}((0, x'_\epsilon) + r_\epsilon x) - x_\epsilon|^2} \right)^{\frac{n-2}{2}} \qquad \text{for } x \in B_0(R) \cap \{x_1 < 0\}$$

For any x $\in B_0(R)\cap\{x_1\le 0\}$ we get from Proposition 4.8.1 that as $\epsilon\to 0$

$$\tilde v_\epsilon(x) = (1+o(1))\frac{r_\epsilon^{n-2}}{\mu_\epsilon^{\frac{n-2}{2}}}\left(\left(\frac{k_\epsilon}{k_\epsilon^2+\frac{|\mathcal{T}((0,x'_\epsilon)+r_\epsilon x)-x_\epsilon|^2}{n(n-2)}}\right)^{\frac{n-2}{2}} - \left(\frac{k_\epsilon}{k_\epsilon^2+\frac{\left|\mathcal{T}((0,x'_\epsilon)+r_\epsilon x)-\pi_{\mathcal{T}}^{-1}(x_\epsilon)\right|^2}{n(n-2)}}\right)^{\frac{n-2}{2}}\right)$$

(4.97)

$$= (1+o(1))\left(\frac{k_\epsilon}{\mu_\epsilon}\right)^{\frac{n-2}{2}}\left(\left(\frac{1}{\left(\frac{k_\epsilon}{r_\epsilon}\right)^2+\frac{|\mathcal{T}((0,x'_\epsilon)+r_\epsilon x)-x_\epsilon|^2}{n(n-2)r_\epsilon^2}}\right)^{\frac{n-2}{2}} - \left(\frac{1}{\left(\frac{k_\epsilon}{r_\epsilon}\right)^2+\frac{\left|\mathcal{T}((0,x'_\epsilon)+r_\epsilon x)-\pi_{\mathcal{T}}^{-1}(x_\epsilon)\right|^2}{n(n-2)r_\epsilon^2}}\right)^{\frac{n-2}{2}}\right)$$

Fom the properties of the boundary map $\mathcal{T}$ one obtains that for any x $\in B_0(R)\cap\{x_1\le 0\}$:

$$\left|\frac{\mathcal{T}((0,x'_\epsilon)+r_\epsilon x)-x_\epsilon}{r_\epsilon}\right| = \left|\frac{\mathcal{T}((0,x'_\epsilon)+r_\epsilon x)-\mathcal{T}((0,x'_\epsilon)+r_\epsilon((x_\epsilon)_1/r_\epsilon,0))}{r_\epsilon}\right|$$
$$= (1+o(1))\,|x-((x_\epsilon)_1/r_\epsilon,0)| \qquad \text{as } \epsilon\to 0$$

and

$$\left|\frac{\mathcal{T}((0,x'_\epsilon)+r_\epsilon x)-\pi_{\mathcal{T}}(x_\epsilon)}{r_\epsilon}\right| = \left|\frac{\mathcal{T}((0,x'_\epsilon)+r_\epsilon x)-\mathcal{T}((0,x'_\epsilon)+r_\epsilon(-(x_\epsilon)_1/r_\epsilon,0))}{r_\epsilon}\right|$$
$$= (1+o(1))\,|x+((x_\epsilon)_1/r_\epsilon,0)| \qquad \text{as } \epsilon\to 0$$

Recall that in Theorem 4.4 we have obtained $\lim\limits_{\epsilon\to 0}\frac{k_\epsilon}{\mu_\epsilon}=1$ and $\lim\limits_{\epsilon\to 0}\frac{k_\epsilon}{r_\epsilon}=0$. Passing to limits as $\epsilon\to 0$ in (4.97), we then have the following pointwise convergence.

(4.98)

$$\lim_{\epsilon\to 0}\tilde v_\epsilon(x) = \frac{(n(n-2))^{\frac{n-2}{2}}}{|x-(1,0)|^{n-2}} - \frac{(n(n-2))^{\frac{n-2}{2}}}{|x+(1,0)|^{n-2}} \qquad \text{for } x\in(B_0(R)\setminus\{(1,0)\})\cap\{x_1\le 0\}$$

For $i,j=1,\dots,n$, we let $(\tilde g_\epsilon)_{ij}(x) = (\partial_i\mathcal{T}((0,x'_\epsilon)+r_\epsilon x),\partial_j\mathcal{T}((0,x'_\epsilon)+r_\epsilon x))$, the induced metric on the domain $B_0(R)\cap\{x_1<0\}$, and let Δ_g denote the Laplace-Beltrami operator with respect to the metric g. From eqn (4.4) it follows that given any $R>0$, $\tilde v_\epsilon$ weakly satisfies the following equation for $\epsilon>0$ sufficiently small

$$(4.99)\quad \begin{cases} \Delta_{\tilde g_\epsilon}\tilde v_\epsilon + r_\epsilon^2\,(a\circ\mathcal{T}((0,x'_\epsilon)+r_\epsilon x))\,\tilde v_\epsilon = \left(\frac{\mu_\epsilon}{r_\epsilon}\right)^{2-s_\epsilon}\frac{\tilde v_\epsilon^{2^*(s_\epsilon)-1}}{\left|\frac{\mathcal{T}((0,x'_\epsilon)+r_\epsilon x)}{r_\epsilon}\right|^{s_\epsilon}} & \text{in } B_0(R)\cap\{x_1<0\}\\ \tilde v_\epsilon = 0 & \text{on } B_0(R)\cap\{x_1=0\}\end{cases}$$

Let $D\subset\subset\overline{\mathbb{R}^n_-}\setminus\{\theta_0\}$ be an open set with compact closure. From(4.98) it follows that there exists a constant $C_D>0$ such that for all $\epsilon>0$ sufficiently small

$$0\le\tilde v_\epsilon(x)\le C_D \qquad \text{for all } x\in D$$

Again from the properties of the boundary chart $\mathcal{T}$, it follows that for any $p > 1$ there exists a constant $C'_D > 0$ such that

$$\int_{D\cap\{x_1<0\}} \left[\frac{(\tilde{v}_\epsilon)^{2^*(s_\epsilon)-1}}{\left|\frac{\mathcal{T}((0,x'_\epsilon)+r_\epsilon x)}{r_\epsilon}\right|^{s_\epsilon}}\right]^p dx \leq C'_D \int_{D\cap\{x_1<0\}} \frac{1}{\left|\frac{(0,x'_\epsilon)}{r_\epsilon}+x\right|^{s_\epsilon p}}\, dx$$
$$\leq C'_D \int_{B_0(R)} \frac{1}{\left|\frac{(0,x'_\epsilon)}{r_\epsilon}+x\right|^{s_\epsilon p}}\, dx$$

Choosing $s_\epsilon > 0$ sufficiently small it follows that the right hand side of equation (4.63) is uniformly bounded in $L^p(D)$ for some $p > n$. Then from standard elliptic estimates (see for instance [**14**]) it follows that for any $D' \subset\subset D$ $\|\tilde{v}_\epsilon\|_{C^{1,\alpha}(D')} = O(1)$ as $\epsilon \to 0$, $\alpha > 0$ and $\tilde{v}_\epsilon$ vanishes on the boundary $D' \cap \{x_1 = 0\}$. Hence the sequence $(\tilde{v}_\epsilon)_{\epsilon>0}$ is precompact in $C^1(\overline{D'})$. From (4.98) it therefore follows that

$$\lim_{\epsilon\to 0} \tilde{v}_\epsilon \longrightarrow \frac{(n(n-2))^{\frac{n-2}{2}}}{|x-(1,0)|^{n-2}} - \frac{(n(n-2))^{\frac{n-2}{2}}}{|x+(1,0)|^{n-2}} \qquad \text{in } C^1(\Omega')$$

This completes the proof of Proposition 4.8.2. □

4.8.2. Estimates on the blow up rates: The Boundary Case.

Suppose that the sequence of blow up points $(x_\epsilon)_{\epsilon>0}$ converges to a point on the boundary, i.e suppose

$$\lim_{\epsilon\to 0} x_\epsilon = x_0 \in \partial\Omega \tag{4.100}$$

We let

$$r_\epsilon = d(x_\epsilon, \partial\Omega) \tag{4.101}$$

Then

$$\lim_{\epsilon\to 0} r_\epsilon = 0$$

and from (4.25), we have as $\epsilon \to 0$

$$\mu_\epsilon = o(r_\epsilon) \qquad \text{and} \qquad k_\epsilon = o(r_\epsilon)$$

As before, let $\mathcal{T}$ be a parametrisation of the boundary $\partial\Omega$ as in (4.18) around the point $p = x_0$. We shall apply the Pohozaev identity for the Hardy Sobolev equation to the domain $\mathcal{T}\left(B_{\mathcal{T}^{-1}(x_\epsilon)}(r_\epsilon/2)\right)$. Note that since $\frac{d(x_\epsilon,\partial\Omega)}{r_\epsilon} = 1$ for all $\epsilon > 0$, so $\overline{B_{\mathcal{T}^{-1}(x_\epsilon)}(r_\epsilon/2)} \subset \mathbb{R}^n_-$ for $\epsilon > 0$ small, and therefore $\mathcal{T}\left(B_{\mathcal{T}^{-1}(x_\epsilon)}(r_\epsilon/2)\right) \subset \Omega$ for $\epsilon > 0$ small. The Pohozaev identity (4.87) gives us

$$\int_{\mathcal{T}\left(B_{\mathcal{T}^{-1}(x_\epsilon)}(r_\epsilon/2)\right)} \left(a + \frac{(x-x_\epsilon,\nabla a)}{2}\right) u_\epsilon^2\, dx - \frac{s_\epsilon(n-2)}{2(n-s_\epsilon)} \int_{\mathcal{T}\left(B_{\mathcal{T}^{-1}(x_\epsilon)}(r_\epsilon/2)\right)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \frac{(x,x_\epsilon)}{|x|^2}\, dx =$$

$$\int_{\partial\left(\mathcal{T}\left(B_{\mathcal{T}^{-1}(x_\epsilon)}(r_\epsilon/2)\right)\right)} (x-x_\epsilon,\nu)\left(\frac{|\nabla u_\epsilon|^2}{2} + \frac{a u_\epsilon^2}{2} - \frac{1}{2^*(s_\epsilon)}\frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\right) - \left((x-x_\epsilon,\nabla u_\epsilon) + \frac{n-2}{2}u_\epsilon\right)\partial_\nu u_\epsilon\, d\sigma \tag{4.102}$$

for all $\epsilon > 0$ small. We will estimate each of the terms in the integral above and calculate the limit as $\epsilon \to 0$. We obtain

Theorem 4.10. *Let Ω be a bounded smooth oriented domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$, and let $a \in C^1(\overline{\Omega})$ be such that the operator $\Delta + a$ is coercive in Ω. Let $(s_\epsilon)_{\epsilon>0} \in (0,2)$ be a sequence such that $\lim_{\epsilon\to 0} s_\epsilon = 0$. Suppose that the sequence $(u_\epsilon)_{\epsilon>0} \in H^2_{1,0}(\Omega)$, where for each $\epsilon > 0$, u_ϵ satisfies (4.4) and (4.5), is a* blowup sequence, *i.e*

$$u_\epsilon \rightharpoonup 0 \qquad \textit{weakly in } H^2_{1,0}(\Omega) \qquad \textit{as } \epsilon \to 0$$

We let $(\mu_\epsilon)_\epsilon \in (0,+\infty)$ and $(x_\epsilon)_\epsilon \in \Omega$ be such that

$$\mu_\epsilon^{-\frac{n-2}{2}} = u_\epsilon(x_\epsilon) = \max_{x\in\Omega} u_\epsilon(x).$$

Assume that (4.100) and (4.101) hold. Then

(1) *If $n = 3$ or $a \equiv 0$, then as $\epsilon \to 0$*

$$\lim_{\epsilon\to 0} \frac{s_\epsilon r_\epsilon^{n-2}}{\mu_\epsilon^{n-2}} = \frac{n^{n-1}(n-2)^{n-1}K(n,0)^{n/2}\omega_{n-1}}{2^{n-2}}.$$

Moreover, $d(x_\epsilon, \partial\Omega) = (1 + o(1))|x_\epsilon|$ as $\epsilon \to 0$. In particular $x_0 = 0$.

(2) *If $n = 4$. Then as $\epsilon \to 0$*

$$\frac{s_\epsilon}{4}\left(K(4,0)^{-2} + o(1)\right) - \left(\frac{\mu_\epsilon}{r_\epsilon}\right)^2 (32\omega_3 + o(1)) = \mu_\epsilon^2 \log\left(\frac{r_\epsilon}{\mu_\epsilon}\right)[d_4 a(x_0) + o(1)]$$

and

$$s\left(1 - \left(\frac{r_\epsilon}{|x_\epsilon|}\right)^2 + o(1)\right) = \mu_\epsilon^2 \log\left(\frac{r_\epsilon}{\mu_\epsilon}\right)\left[4d_4K(4,0)^2 a(x_0) + o(1)\right]$$

(3) *If $n \geq 5$. Then as $\epsilon \to 0$*

$$\frac{s_\epsilon(n-2)}{2n}\left(K(n,0)^{-n/2} + o(1)\right) - \left(\frac{\mu_\epsilon}{r_\epsilon}\right)^{n-2}\left(\frac{n^{n-2}(n-2)^n\omega_{n-1}}{2^{n-1}} + o(1)\right) = \mu_\epsilon^2\left[d_n a(x_0) + o(1)\right]$$

and

$$s\left(1 - \left(\frac{r_\epsilon}{|x_\epsilon|}\right)^2 + o(1)\right) = \mu_\epsilon^2\left[\frac{2n}{n-2}d_n K(n,0)^2 a(x_0) + o(1)\right]$$

where

$$d_n = \int_{\mathbb{R}^n} \frac{1}{\left(1 + \frac{|x|^2}{n(n-2)}\right)^{n-2}}\,dx \quad \textit{for } n \geq 5 \qquad \textit{and } d_4 = 64\omega_3$$

Proof. For convenience we define

$$F_\epsilon = (x - x_\epsilon, \nu)\left(\frac{|\nabla u_\epsilon|^2}{2} + \frac{a u_\epsilon^2}{2} - \frac{1}{2^*(s_\epsilon)}\frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\right) - \left((x - x_\epsilon, \nabla u_\epsilon) + \frac{n-2}{2}u_\epsilon\right)\partial_\nu u_\epsilon$$

Step 1: We claim that

$$\left(\frac{\mu_\epsilon}{r_\epsilon}\right)^{2-n} \int_{\partial\left(\mathcal{T}\left(B_{\mathcal{T}^{-1}(x_\epsilon)}(r_\epsilon/2)\right)\right)} F_\epsilon\,d\sigma = -\frac{n^{n-2}(n-2)^n\omega_{n-1}}{2^{n-1}} + o(1) \qquad \text{as } \epsilon \to 0$$

PROOF. We write $\mathcal{T}^{-1}(x_\epsilon) = ((x_\epsilon)_1, x'_\epsilon)$. For $\epsilon > 0$, let

$$\tilde{v}_\epsilon(x) := \frac{r_\epsilon^{n-2}}{\mu_\epsilon^{\frac{n-2}{2}}} u_\epsilon \circ \mathcal{T}((0, x'_\epsilon) + r_\epsilon x) \qquad \text{for } \ x \in \frac{U - (0, x'_\epsilon)}{r_\epsilon} \cap \{x_1 \le 0\}$$

Then from Proposition 4.8.2 it follows that

$$\lim_{\epsilon \to 0} \tilde{v}_\epsilon = \frac{(n(n-2))^{\frac{n-2}{2}}}{|x - (-1,0)|^{n-2}} - \frac{(n(n-2))^{\frac{n-2}{2}}}{|x - (1,0)|^{n-2}} \qquad in\ C^1_{loc}(\overline{\mathbb{R}^n_-} \setminus \{(-1,0)\})$$

For simplicity we write for $x \in \overline{\mathbb{R}^n_-} \setminus \{(-1,0)\}$

$$\frac{(n(n-2))^{\frac{n-2}{2}}}{|x - (-1,0)|^{n-2}} - \frac{(n(n-2))^{\frac{n-2}{2}}}{|x - (1,0)|^{n-2}} := \tilde{v}(x)$$

We define the rescaled metric

$$\tilde{g}_\epsilon(x) = \mathcal{T}^* g\left((0, x'_\epsilon) + r_\epsilon x)\right) \qquad \text{for } \ x \in \frac{U - (0, x'_\epsilon)}{r_\epsilon} \cap \{x_1 \le 0\}$$

With the change of variable $x \mapsto \mathcal{T}\left((0, x'_\epsilon) + r_\epsilon z\right)$ we obatin

$$\left(\frac{\mu_\epsilon}{r_\epsilon}\right)^{2-n} \int\limits_{\partial\left(\mathcal{T}\left(B_{\mathcal{T}^{-1}(x_\epsilon)}(r_\epsilon/2)\right)\right)} F_\epsilon \, d\sigma = \left(\frac{\mu_\epsilon}{r_\epsilon}\right)^{2-n} \int\limits_{\mathcal{T}\left(\partial B_{((x_\epsilon)_1, x'_\epsilon)}(r_\epsilon/2)\right)} F_\epsilon \, d\sigma =$$

$$\int\limits_{\partial B_{\left(\frac{(x_\epsilon)_1}{r_\epsilon},0\right)}(1/2)} \left(\frac{\mathcal{T}\left((0,x'_\epsilon) + r_\epsilon z\right) - \mathcal{T}\left((0,x'_\epsilon) + r_\epsilon\left(\frac{(x_\epsilon)_1}{r_\epsilon},0\right)\right)}{r_\epsilon}, \nu\right)_{\tilde{g}_\epsilon} \frac{|\nabla \tilde{v}_\epsilon|^2}{2} \, d\sigma +$$

$$\int\limits_{\partial B_{\left(\frac{(x_\epsilon)_1}{r_\epsilon},0\right)}(1/2)} \left(\frac{\mathcal{T}\left((0,x'_\epsilon) + r_\epsilon z\right) - \mathcal{T}\left((0,x'_\epsilon) + r_\epsilon\left(\frac{(x_\epsilon)_1}{r_\epsilon},0\right)\right)}{r_\epsilon}, \nu\right)_{\tilde{g}_\epsilon} r_\epsilon^2 a\left(\mathcal{T}\left((0,x'_\epsilon) + r_\epsilon z\right)\right) \frac{\tilde{v}_\epsilon^2}{2} \, d\sigma -$$

$$\int\limits_{\partial B_{\left(\frac{(x_\epsilon)_1}{r_\epsilon},0\right)}(1/2)} \left(\frac{\mathcal{T}\left((0,x'_\epsilon) + r_\epsilon z\right) - \mathcal{T}\left((0,x'_\epsilon) + r_\epsilon\left(\frac{(x_\epsilon)_1}{r_\epsilon},0\right)\right)}{r_\epsilon}, \nu\right)_{\tilde{g}_\epsilon} \frac{1}{2^*(s_\epsilon)} \left(\frac{\mu_\epsilon}{r_\epsilon}\right)^{2-s_\epsilon} \frac{\tilde{v}_\epsilon^{2^*(s_\epsilon)}}{\left|\frac{\mathcal{T}((0,x'_\epsilon)+r_\epsilon z)}{r_\epsilon}\right|^{s_\epsilon}} \, d\sigma$$

(4.103)

$$- \int\limits_{\partial B_{\left(\frac{(x_\epsilon)_1}{r_\epsilon},0\right)}(1/2)} \left(\left(\frac{\mathcal{T}\left((0,x'_\epsilon) + r_\epsilon z\right) - \mathcal{T}\left((0,x'_\epsilon) + r_\epsilon\left(\frac{(x_\epsilon)_1}{r_\epsilon},0\right)\right)}{r_\epsilon}, \nabla \tilde{v}_\epsilon\right)_{\tilde{g}_\epsilon} + \frac{n-2}{2}\tilde{v}_\epsilon\right) \partial_\nu \tilde{v}_\epsilon \, d\sigma$$

Since $D_0\mathcal{T} = \mathbb{I}_{\mathbb{R}^n}$ we have

$$d(x_\epsilon, \partial\Omega) = (1 + o(1))\, |(x_\epsilon)_1| \qquad as\ \epsilon \to 0$$

So

$$\lim_{\epsilon \to 0} \left(\frac{(x_\epsilon)_1}{r_\epsilon}, 0\right) = (-1, 0) \in \mathbb{R}^n_-$$

And it further follows that one always has for ϵ small

$$\frac{1}{\left|\frac{\mathcal{T}((0,x'_\epsilon)+r_\epsilon z)}{r_\epsilon}\right|^{s_\epsilon}} \leq 2 \qquad \text{for } \; z \in \partial B_{\left(\frac{(x_\epsilon)_1}{r_\epsilon},0\right)}(1/2)$$

Passing to limts as $\epsilon \to 0$ in (4.103), using Proposition 4.8.2, we get as $\epsilon \to 0$

$$\left(\frac{\mu_\epsilon}{r_\epsilon}\right)^{2-n} \int_{\partial\left(\mathcal{T}\left(B_{\mathcal{T}^{-1}(x_\epsilon)}(r_\epsilon/2)\right)\right)} F_\epsilon \, d\sigma =$$
$$\int_{\partial \; B_{(-1,0)}(1/2)} \left((z-(-1,0),\nu)\frac{|\nabla\tilde{v}|^2}{2} - \left((z-(-1,0),\nabla\tilde{v}) + \frac{n-2}{2}\tilde{v}\right)\partial_\nu\tilde{v}\right) \, d\sigma + o(1)$$

Let $0 < \delta < 1/2$. Since $\Delta\tilde{v} = 0$ in $B_{(-1,0)}(1/2) \setminus B_{(-1,0)}(\delta)$, applying the Pohozaev identity (4.86) we have that

$$\int_{\partial \; B_{(-1,0)}(1/2)} \left((z-(-1,0),\nu)\frac{|\nabla\tilde{v}|^2}{2} - \left((z-(-1,0),\nabla\tilde{v}) + \frac{n-2}{2}\tilde{v}\right)\partial_\nu\tilde{v}\right) \, d\sigma =$$
$$\int_{\partial \; B_{(-1,0)}(\delta)} \left((z-(-1,0),\nu)\frac{|\nabla\tilde{v}|^2}{2} - \left((z-(-1,0),\nabla\tilde{v}) + \frac{n-2}{2}\tilde{v}\right)\partial_\nu\tilde{v}\right) \, d\sigma$$

and so the map

$$\delta \mapsto \int_{\partial \; B_{(-1,0)}(\delta)} \left((z-(-1,0),\nu)\frac{|\nabla\tilde{v}|^2}{2} - \left((z-(-1,0),\nabla\tilde{v}) + \frac{n-2}{2}\tilde{v}\right)\partial_\nu\tilde{v}\right) \, d\sigma$$

is constant on $(0, 1/2]$. We write for $x \in \overline{\mathbb{R}^n_-} \setminus \{(-1,0)\}$

$$\begin{aligned}\tilde{v}(x) &= \frac{(n(n-2))^{\frac{n-2}{2}}}{|x-(-1,0)|^{n-2}} - \frac{(n(n-2))^{\frac{n-2}{2}}}{|x-(1,0)|^{n-2}} \\ &= \frac{(n(n-2))^{\frac{n-2}{2}}}{|x-(-1,0)|^{n-2}} + h(x)\end{aligned}$$

where $h(x) = -\frac{(n(n-2))^{\frac{n-2}{2}}}{|x-(1,0)|^{n-2}}$ is the harmonic part of $\tilde{v}$. We let $\theta_0 = (-1,0)$, $\tilde{v_1}(x) := \frac{1}{n^{\frac{n-2}{2}}(n-2)^{\frac{n}{2}}\omega_{n-1}}\tilde{v}(x)$ and $g(x) := \frac{1}{n^{\frac{n-2}{2}}(n-2)^{\frac{n}{2}}\omega_{n-1}}h(x)$. Then one has

$$
\begin{aligned}
\tilde{v_1}(x) &= \frac{1}{(n-2)\omega_{n-1}}\frac{1}{|x-\theta_0|^{n-2}} + g(x),\\
\partial_j\tilde{v_1}(x) &= -\frac{1}{\omega_{n-1}}\frac{1}{|x-\theta_0|^{n-1}}\frac{(x-\theta_0)^j}{|x-\theta_0|} + \partial_j g(x) \qquad \text{for } 1\le j\le n,\\
|\nabla\tilde{v_1}(x)|^2 &= \frac{1}{\omega_{n-1}^2}\frac{1}{|x-\theta_0|^{2n-2}} - \frac{1}{\omega_{n-1}}\frac{2}{|x-\theta_0|^{n-1}}\frac{(x-\theta_0,\nabla g)}{|x-\theta_0|} + |\nabla g(x)|^2,\\
(x-\theta_0,\nabla\tilde{v_1}(x)) &= -\frac{1}{\omega_{n-1}}\frac{1}{|x-\theta_0|^{n-2}} + (x-\theta_0,\nabla g(x)),\\
(x-\theta_0,\nabla\tilde{v_1}(x))^2 &= \frac{1}{\omega_{n-1}^2}\frac{1}{|x-\theta_0|^{2n-4}} - \frac{1}{\omega_{n-1}}\frac{2}{|x-\theta_0|^{n-2}}(x-\theta_0,\nabla g(x)) + (x-\theta_0,\nabla g(x))^2,\\
(x-\theta_0,\nabla\tilde{v_1}(x))\ \tilde{v_1}(x) &= -\frac{1}{(n-2)\omega_{n-1}^2}\frac{1}{|x-\theta_0|^{2n-4}} - \frac{1}{\omega_{n-1}}\frac{1}{|x-\theta_0|^{n-2}}g(x)\\
&\quad + (x-\theta_0,\nabla g(x))\frac{1}{(n-2)\omega_{n-1}}\frac{1}{|x-\theta_0|^{n-2}} + (x-\theta_0,\nabla g(x))\ g(x)
\end{aligned}
$$

So, noting that $\partial_\nu\tilde{v}(z) = \frac{(z-\theta_0,\nabla\tilde{v})}{\delta}$ on $\partial\ B_{\theta_0}(\delta)$, we obtain

$$
\begin{aligned}
&\frac{1}{n^{n-2}(n-2)^n\omega_{n-1}^2}\int\limits_{\partial\ B_{\theta_0}(\delta)}\left(\delta\frac{|\nabla\tilde{v}|^2}{2} - \frac{(z-\theta_0,\nabla\tilde{v})^2}{\delta} - \frac{n-2}{2}\frac{(z-\theta_0,\nabla\tilde{v})}{\delta}\tilde{v}\right)\,d\sigma\\
&= \int\limits_{\partial B_{\theta_0}(\delta)}\frac{1}{\omega_{n-1}^2}\frac{1}{2\delta^{2n-3}} - \frac{1}{\omega_{n-1}}\frac{1}{\delta^{n-1}}(x-\theta_0,\nabla g(x)) + \delta\frac{|\nabla g(x)|^2}{2}\ d\sigma\\
&+ \int\limits_{\partial B_{\theta_0}(\delta)} -\frac{1}{\omega_{n-1}^2}\frac{1}{\delta^{2n-3}} + \frac{1}{\omega_{n-1}}\frac{2}{\delta^{n-1}}(x-\theta_0,\nabla g(x)) - \frac{(x-\theta_0,\nabla g(x))^2}{\delta}\ d\sigma\\
&+ \int\limits_{\partial B_{\theta_0}(\delta)}\frac{1}{\omega_{n-1}^2}\frac{1}{2\delta^{2n-3}} + \frac{1}{\omega_{n-1}}\frac{(n-2)}{2\delta^{n-1}}g(x) - \frac{1}{\omega_{n-1}}\frac{1}{2\delta^{n-1}}(x-\theta_0,\nabla g(x))\\
&- \frac{n-2}{2\delta}(x-x_0,\nabla g(x))\ g(x)\ d\sigma\\
&= \int\limits_{\partial B_{\theta_0}(\delta)}\frac{1}{\omega_{n-1}}\frac{(n-2)}{2\delta^{n-1}}g(x)\ d\sigma + \int\limits_{\partial B_{\theta_0}(\delta)}\frac{1}{\omega_{n-1}}\frac{1}{2\delta^{n-1}}(x-\theta_0,\nabla g(x))\ d\sigma\\
&+ \int\limits_{\partial B_{\theta_0}(\delta)}\left[\delta\frac{|\nabla g(x)|^2}{2} - \frac{(x-\theta_0,\nabla g(x))^2}{\delta} - \frac{n-2}{2\delta}(x-\theta_0,\nabla g(x))\ g(x)\right]\ d\sigma
\end{aligned}
$$

Therefore we have

$$\frac{1}{n^{n-2}(n-2)^n\omega_{n-1}^2}\lim_{\delta\to 0}\int_{\partial\, B_{\theta_0}(\delta)}\left(\delta\frac{|\nabla\tilde{v}|^2}{2}-\frac{(z-\theta_0,\nabla\tilde{v})^2}{\delta}-\frac{n-2}{2}\frac{(z-\theta_0,\nabla\tilde{v})}{\delta}\tilde{v}\right)\,d\sigma$$

$$=\lim_{\delta\to 0}\int_{\partial B_{\theta_0}(\delta)}\frac{1}{\omega_{n-1}}\frac{(n-2)}{2\delta^{n-1}}g(x)\,d\sigma+\lim_{\delta\to 0}\int_{\partial B_{\theta_0}(\delta)}\frac{1}{\omega_{n-1}}\frac{1}{2\delta^{n-1}}(x-\theta_0,\nabla g(x))\,d\sigma$$

$$+\lim_{\delta\to 0}\int_{\partial B_{\theta_0}(\delta)}\left[\delta\frac{|\nabla g(x)|^2}{2}-\frac{(x-\theta_0,\nabla g(x))^2}{\delta}-\frac{n-2}{2\delta}(x-\theta_0,\nabla g(x))\,g(x)\right]\,d\sigma$$

$$=\frac{(n-2)}{2}g(\theta_0)$$

And

$$\lim_{\delta\to 0}\int_{\partial\, B_{\theta_0}(\delta)}\left(\delta\frac{|\nabla\tilde{v}|^2}{2}-\frac{(z-\theta_0,\nabla\tilde{v})^2}{\delta}-\frac{n-2}{2}\frac{(z-\theta_0,\nabla\tilde{v})}{\delta}\tilde{v}\right)\,d\sigma$$

$$=n^{\frac{n-2}{2}}(n-2)^{\frac{n}{2}}\omega_{n-1}\frac{(n-2)}{2}h(\theta_0)=-\frac{n^{n-2}(n-2)^n\omega_{n-1}}{2^{n-1}}$$

Hence

$$\int_{\partial\, B_{(-1,0)}(1/2)}\left((z-(-1,0),\nu)\,\frac{|\nabla\tilde{v}|^2}{2}-\left((z-(-1,0),\nabla\tilde{v})+\frac{n-2}{2}\tilde{v}\right)\partial_\nu\tilde{v}\right)\,d\sigma$$

$$=-\frac{n^{n-2}(n-2)^n\omega_{n-1}}{2^{n-1}}$$

This completes Step 1. □

Step 2: We claim that

$$\int_{\mathcal{T}\left(B_{\mathcal{T}^{-1}(x_\epsilon)}(r_\epsilon/2)\right)}\frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\frac{(x,x_\epsilon)}{|x|^2}\,dx=\left(\frac{1}{K(n,0)}\right)^{\frac{2^*}{2^*-2}}+o(1)\qquad\text{as }\ \epsilon\to 0$$

Proof. Since $D_0\mathcal{T}=\mathbb{I}_{\mathbb{R}^n}$, so for $\epsilon>0$ sufficiently small one has

$$B_{x_\epsilon}(r_\epsilon/4)\subset\mathcal{T}\left(B_{\mathcal{T}^{-1}(x_\epsilon)}(r_\epsilon/2)\right)\subset B_{x_\epsilon}(3r_\epsilon/4)$$

And hence for $\epsilon>0$ sufficiently small and since the integrand is nonnegative (which is a consequence of the computations below)

$$\int_{B_{x_\epsilon}(r_\epsilon/4)}\frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\frac{(x,x_\epsilon)}{|x|^2}\,dx\le\int_{\mathcal{T}\left(B_{\mathcal{T}^{-1}(x_\epsilon)}(r_\epsilon/2)\right)}\frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\frac{(x,x_\epsilon)}{|x|^2}\,dx\le\int_{B_{x_\epsilon}(3r_\epsilon/4)}\frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\frac{(x,x_\epsilon)}{|x|^2}\,dx$$

We fix a $0 < \delta < 1$ and calculate $\int\limits_{B_{x_\epsilon}(r_\epsilon\delta)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \frac{(x,x_\epsilon)}{|x|^2}\, dx$. Recall our definition of v_ϵ in Theorem 4.4. With a change of variable we obtain

(4.104)
$$\int\limits_{B_{x_\epsilon}(r_\epsilon\delta)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \frac{(x,x_\epsilon)}{|x|^2}\, dx = \left(\frac{|x_\epsilon|^{s_\epsilon}}{\mu_\epsilon^{s_\epsilon}}\right)^{\frac{n-2}{2}} \int\limits_{B_0(\delta r_\epsilon/k_\epsilon)} \frac{\left(\frac{x_\epsilon}{|x_\epsilon|}+\frac{k_\epsilon}{|x_\epsilon|}x, \frac{x_\epsilon}{|x_\epsilon|}\right)}{|\frac{x_\epsilon}{|x_\epsilon|}+\frac{k_\epsilon}{|x_\epsilon|}x|^2} \frac{v_\epsilon(x)^{2^*(s_\epsilon)}}{\left|\frac{x_\epsilon}{|x_\epsilon|}+\frac{k_\epsilon}{|x_\epsilon|}x\right|^{s_\epsilon}}\, dx$$

We have obtained earlier in Theorem 4.4 that $\lim\limits_{\epsilon\to 0} \frac{\mu_\epsilon^{s_\epsilon}}{|x_\epsilon|^{s_\epsilon}} = 1$, $\lim\limits_{\epsilon\to 0} \frac{r_\epsilon}{k_\epsilon} = +\infty$ and $\lim\limits_{\epsilon\to 0} \frac{k_\epsilon}{|x_\epsilon|} = 0$ Also we have for all $x \in B_0(\delta r_\epsilon/k_\epsilon)$

$$1 = \left|\frac{x_\epsilon}{|x_\epsilon|}\right| \le \left|\frac{x_\epsilon}{|x_\epsilon|}+\frac{k_\epsilon}{|x_\epsilon|}x\right| + \frac{k_\epsilon}{|x_\epsilon|}|x| \le \left|\frac{x_\epsilon}{|x_\epsilon|}+\frac{k_\epsilon}{|x_\epsilon|}x\right| + \frac{r_\epsilon}{|x_\epsilon|}\delta \le \left|\frac{x_\epsilon}{|x_\epsilon|}+\frac{k_\epsilon}{|x_\epsilon|}x\right| + \delta$$

So for all $x \in B_0(\delta r_\epsilon/k_\epsilon)$

$$\left|\frac{x_\epsilon}{|x_\epsilon|}+\frac{k_\epsilon}{|x_\epsilon|}x\right| \ge 1-\delta$$

Then passing to limits in (4.104), using Theorem 4.4 and the pointwise control of Theorem 4.6, we obtain by Lebesgue dominated convergence theorem

$$\lim_{\epsilon\to 0} \int\limits_{B_{x_\epsilon}(r_\epsilon\delta)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \frac{(x,x_\epsilon)}{|x|^2}\, dx = \int\limits_{\mathbb{R}^n} v^{2^*}\, dx = \left(\frac{1}{K(n,0)}\right)^{\frac{2^*}{2^*-2}}$$

And therefore

$$\int\limits_{\mathcal{T}\left(B_{\mathcal{T}^{-1}(x_\epsilon)}(r_\epsilon/2)\right)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \frac{(x,x_\epsilon)}{|x|^2}\, dx = \left(\frac{1}{K(n,0)}\right)^{\frac{2^*}{2^*-2}} + o(1) \qquad \text{as } \epsilon\to 0$$

This ends Step 2. □

Step 3: We claim that, as $\epsilon \to 0$,

$$\int\limits_{\mathcal{T}\left(B_{\mathcal{T}^{-1}(x_\epsilon)}(r_\epsilon/2)\right)} \left(a + \frac{(x-x_\epsilon, \nabla a)}{2}\right) u_\epsilon^2\, dx = \begin{cases} O(\mu_\epsilon) & \text{for } n=3 \text{ or } a\equiv 0, \\ \mu_\epsilon^2 \log\left(\frac{r_\epsilon}{k_\epsilon}\right)[64\omega_3 a(x_0)+o(1)] & \text{for } n=4, \\ \mu_\epsilon^2[d_n a(x_0)+o(1)] & \text{for } n\ge 5. \end{cases}$$

where

$$d_n = \int\limits_{\mathbb{R}^n} \frac{1}{\left(1+\frac{|x|^2}{n(n-2)}\right)^{n-2}}\, dx \qquad \text{for } n \ge 5$$

Proof. We divide the proof in three steps.

Case 3.1: we assume that $n = 3$. In Theorem 4.6 we have obtained that there exists a constant $C > 0$ such that for all $x \in \Omega$, $\mu_\epsilon^{-1/2} u_\epsilon(x) \le \frac{C}{|x-x_\epsilon|}$ for all $\epsilon > 0$. So we obtain

$$\int\limits_{\mathcal{T}\left(B_{\mathcal{T}^{-1}(x_\epsilon)}(r_\epsilon/2)\right)} \left(a + \frac{(x-x_\epsilon, \nabla a)}{2}\right) u_\epsilon^2\, dx = O(\mu_\epsilon) \int\limits_{\Omega} \frac{1}{|x|^2}\, dx = O(\mu_\epsilon)$$

Case 3.2: we assume that $n = 4$. Since $D_0\mathcal{T} = \mathbb{I}_{\mathbb{R}^n}$, so for $\epsilon > 0$ sufficiently small one has

$$B_{x_\epsilon}(r_\epsilon/4) \subset \mathcal{T}\left(B_{\mathcal{T}^{-1}(x_\epsilon)}(r_\epsilon/2)\right) \subset B_{x_\epsilon}(3r_\epsilon/4)$$

We fix $0 < \delta < 1$ and calculate the following integral. Recall our definition of v_ϵ in Theorem 4.4. With a change of variable we obtain

$$\frac{\mu_\epsilon^{-2}}{\log(r_\epsilon/k_\epsilon)} \int_{B_{x_\epsilon}(\delta r_\epsilon)} \left(a + \frac{(x - x_\epsilon, \nabla a)}{2}\right) u_\epsilon^2 \, dx =$$

$$\frac{1}{\log(r_\epsilon/k_\epsilon)} \left(\frac{k_\epsilon}{\mu_\epsilon}\right)^4 \int_{B_0(\delta r_\epsilon/k_\epsilon)} \left(a(x_\epsilon + k_\epsilon x) + \frac{(k_\epsilon x, \nabla a(x_\epsilon + k_\epsilon x))}{2}\right) v_\epsilon^2 \, dx$$

We have that $\lim\limits_{\epsilon\to 0} \frac{k_\epsilon}{\mu_\epsilon} = 1$ and from Theorem 4.6, it follows that there exists a constant $C > 0$ such that as $\epsilon \to 0$

$$v_\epsilon(x) \leq C \frac{1}{1 + \left(\frac{k_\epsilon}{\mu_\epsilon}\right)^2 |x|^2} \leq C \frac{1}{1 + |x|^2}$$

and therefore

$$\lim_{\epsilon\to 0} \left[\frac{\mu_\epsilon^{-2}}{\log(r_\epsilon/k_\epsilon)} \int_{B_{x_\epsilon}(\delta r_\epsilon)} \left(a + \frac{(x - x_\epsilon, \nabla a)}{2}\right) u_\epsilon^2 \, dx\right] = 64\omega_3 \; a(x_0)$$

And hence

$$\lim_{\epsilon\to 0} \left[\frac{\mu_\epsilon^{-2}}{\log(r_\epsilon/k_\epsilon)} \int_{\mathcal{T}\left(B_{\mathcal{T}^{-1}(x_\epsilon)}(r_\epsilon/2)\right)} \left(a + \frac{(x - x_\epsilon, \nabla a)}{2}\right) u_\epsilon^2 \, dx\right] = 64\omega_3 \; a(x_0)$$

Case 3.2: We assume that $n \geq 5$. Since $D_0\mathcal{T} = \mathbb{I}_{\mathbb{R}^n}$, so for $\epsilon > 0$ small one has

$$B_{x_\epsilon}(r_\epsilon/4) \subset \mathcal{T}\left(B_{\mathcal{T}^{-1}(x_\epsilon)}(r_\epsilon/2)\right) \subset B_{x_\epsilon}(3r_\epsilon/4)$$

We fix a $0 < \delta < 1$ and calculate the following integral. Recall our definition of v_ϵ in theorem 4.4. With a change of variable we obtain

$$\mu_\epsilon^{-2} \int_{B_{x_\epsilon}(r_\epsilon\delta)} \left(a + \frac{(x - x_\epsilon, \nabla a)}{2}\right) u_\epsilon^2 \, dx = \left(\frac{k_\epsilon}{\mu_\epsilon}\right)^n \int_{B_0(\delta r_\epsilon/k_\epsilon)} \left(a(x_\epsilon + k_\epsilon x) + \frac{(k_\epsilon x, \nabla a(x_\epsilon + k_\epsilon x))}{2}\right) v_\epsilon^2 \, dx$$

We have that $\lim\limits_{\epsilon\to 0} \frac{k_\epsilon}{\mu_\epsilon} = 1$ and from Theorem 4.6, it follows that there exists a constant $C > 0$ such that as $\epsilon \to 0$

$$v_\epsilon(x) \leq C \left(\frac{1}{1 + \left(\frac{k_\epsilon}{\mu_\epsilon}\right)^2 |x|^2}\right)^{\frac{n-2}{2}} \leq C \left(\frac{1}{1 + |x|^2}\right)^{\frac{n-2}{2}}.$$

We have that for $n \geq 5$,

$$d_n = \int_{\mathbb{R}^n} \frac{1}{\left(1 + \frac{|x|^2}{n(n-2)}\right)^{n-2}} \, dx < +\infty \qquad \text{for } n \geq 5$$

Therefore

$$\lim_{\epsilon\to 0}\left[\mu_\epsilon^{-2}\int\limits_{B_0(\delta r_\epsilon/k_\epsilon)}\left(a+\frac{(x-x_\epsilon,\nabla a)}{2}\right)u_\epsilon^2\,dx\right]=d_n\,a(x_0)\qquad\text{for } n\geq 5$$

And hence

$$\lim_{\epsilon\to 0}\left[\mu_\epsilon^{-2}\int\limits_{\mathcal{T}\left(B_{\mathcal{T}^{-1}(x_\epsilon)}(r_\epsilon/2)\right)}\left(a+\frac{(x-x_\epsilon,\nabla a)}{2}\right)u_\epsilon^2\,dx\right]=d_n\,a(x_0)\qquad\text{for } n\geq 5$$

This ends Step 3.3. □

Combining Steps 1 to 3 in the Pohozaev identity (4.102) yields, as $\epsilon\to 0$,

$$\lim_{\epsilon\to 0}\frac{s_\epsilon r_\epsilon^{n-2}}{\mu_\epsilon^{n-2}}=\frac{n^{n-1}(n-2)^{n-1}K(n,0)^{n/2}\omega_{n-1}}{2^{n-2}}\text{ if } n=3 \text{ or } a\equiv 0,$$

$$\frac{s_\epsilon}{4}\left(K(4,0)^{-2}+o(1)\right)-\left(\frac{\mu_\epsilon}{r_\epsilon}\right)^2(32\omega_3+o(1))=\mu_\epsilon^2\log\left(\frac{r_\epsilon}{k_\epsilon}\right)[d_4a(x_0)+o(1)]\text{ if } n=4$$

$$\frac{s_\epsilon(n-2)}{2n}\left(K(n,0)^{-n/2}+o(1)\right)-\left(\frac{\mu_\epsilon}{r_\epsilon}\right)^{n-2}\left(\frac{n^{n-2}(n-2)^n\omega_{n-1}}{2^{n-1}}+o(1)\right)=\mu_\epsilon^2\left[d_na(x_0)+o(1)\right]\text{ if } n\geq 5.$$

To get extra informations, we differentiate the Pohozaev identity (4.87) with respect to the j^{th} variable $(x_\epsilon)_j$ and get

$$\int\limits_{\mathcal{T}\left(B_{\mathcal{T}^{-1}(x_\epsilon)}(r_\epsilon/2)\right)}\frac{\partial_j a}{2}u_\epsilon^2\,dx+\frac{s_\epsilon(n-2)}{2(n-s_\epsilon)}\int\limits_{\mathcal{T}\left(B_{\mathcal{T}^{-1}(x_\epsilon)}(r_\epsilon/2)\right)}\frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\frac{x_j}{|x|^2}\,dx=$$

$$\int\limits_{\partial\left(\mathcal{T}\left(B_{\mathcal{T}^{-1}(x_\epsilon)}(r_\epsilon/2)\right)\right)}\left(\nu_j\left(\frac{|\nabla u_\epsilon|^2}{2}+\frac{au_\epsilon^2}{2}-\frac{1}{2^*(s_\epsilon)}\frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\right)-\partial_j u_\epsilon\partial_\nu u_\epsilon\right)\,d\sigma \tag{4.105}$$

Step 4: We claim that

$$\frac{\mu_\epsilon^{2-n}}{r_\epsilon^{1-n}}\int\limits_{\partial\left(\mathcal{T}\left(B_{\mathcal{T}^{-1}(x_\epsilon)}(r_\epsilon/2)\right)\right)}\left(\nu_1\left(\frac{|\nabla u_\epsilon|^2}{2}+\frac{au_\epsilon^2}{2}-\frac{1}{2^*(s_\epsilon)}\frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\right)-\partial_1 u_\epsilon\partial_\nu u_\epsilon\right)\,d\sigma$$

$$=-\frac{n^{n-2}(n-2)^n\omega_{n-1}}{2^{n-1}}+o(1) \tag{4.106}$$

PROOF. We write $\mathcal{T}^{-1}(x_\epsilon) = ((x_\epsilon)_1, x'_\epsilon)$. Then as Step 1 above, using Proposition 4.8.2 we have as $\epsilon \to 0$

$$\frac{\mu_\epsilon^{2-n}}{r_\epsilon^{1-n}} \int_{\partial\left(\mathcal{T}\left(B_{\mathcal{T}^{-1}(x_\epsilon)}(r_\epsilon/2)\right)\right)} \left(\nu_j \left(\frac{|\nabla u_\epsilon|^2}{2} + \frac{a u_\epsilon^2}{2} - \frac{1}{2^*(s_\epsilon)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}}\right) - \partial_j u_\epsilon \partial_\nu u_\epsilon\right) d\sigma$$
$$= \int_{\partial B_{(-1,0)}(1/2)} \left(\nu_j \frac{|\nabla \tilde{v}|^2}{2} - \partial_j \tilde{v}\ \partial_\nu \tilde{v}\right) d\sigma + o(1)$$

where $\tilde{v}_\epsilon$ and $\tilde{v}$ are as in Step 1 above. In particular

$$\tilde{v}(x) = \frac{(n(n-2))^{\frac{n-2}{2}}}{|x-\theta_0)|^{n-2}} + h(x) \qquad \text{for } x \in \overline{\mathbb{R}^n_-} \setminus \{\theta_0\} \tag{4.107}$$

where $h(x) = -\frac{(n(n-2))^{\frac{n-2}{2}}}{|x-(1,0)|^{n-2}}$. Arguing as in Step 1, we get that

(4.108)
$$\int_{\partial\ B_{(-1,0)}(1/2)} \left(\nu_j \frac{|\nabla \tilde{v}|^2}{2} - \partial_j \tilde{v}\ \partial_\nu \tilde{v}\right) d\sigma = \omega_{n-1}(n-2)(n(n-2))^{\frac{n-2}{2}} \partial_j h(\theta_0)$$

For $j = 1$ we get

$$\int_{\partial\ B_{(-1,0)}(1/2)} \left(\nu_1 \frac{|\nabla \tilde{v}|^2}{2} - \partial_1 \tilde{v}\ \partial_\nu \tilde{v}\right) d\sigma = -\frac{n^{n-2}(n-2)^n \omega_{n-1}}{2^{n-1}} \tag{4.109}$$

This completes Step 4. □

Step 5: We claim that

(4.110)
$$\int_{\mathcal{T}\left(B_{\mathcal{T}^{-1}(x_\epsilon)}(r_\epsilon/2)\right)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \frac{x_1}{|x|^2}\, dx = \frac{(x_\epsilon)_1}{|x_\epsilon|^2} \left(\frac{1}{K(n,0)}\right)^{\frac{2^*}{2^*-2}} (1 + o(1)) \qquad \text{as } \epsilon \to 0$$

PROOF. We proceed as in Step 2 above. We fix a $0 < \delta < 1$ and calculate $|x_\epsilon|^2 \int_{B_{x_\epsilon}(r_\epsilon \delta)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \frac{x_1}{|x|^2}\, dx$. From our definition of v_ϵ in Theorem 4.4 we obtain with a change of variable

$$|x_\epsilon|^2 \int_{B_0(\delta r_\epsilon)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \frac{x_1}{|x|^2}\, dx = \left(\frac{|x_\epsilon|^{s_\epsilon}}{\mu_\epsilon^{s_\epsilon}}\right)^{\frac{n-2}{2}} \int_{B_0(\delta r_\epsilon/k_\epsilon)} \frac{(x_\epsilon)_1 + k_\epsilon x_1}{|\frac{x_\epsilon}{|x_\epsilon|} + \frac{k_\epsilon}{|x_\epsilon|} x|^2} \frac{v_\epsilon(x)^{2^*(s_\epsilon)}}{\left|\frac{x_\epsilon}{|x_\epsilon|} + \frac{k_\epsilon}{|x_\epsilon|} x\right|^{s_\epsilon}}\, dx$$

Then as in Step 2

$$\lim_{\epsilon \to 0} \left[\frac{|x_\epsilon|^2}{(x_\epsilon)_1} \int_{B_0(\delta r_\epsilon)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \frac{x_1}{|x|^2}\, dx\right] = \int_{\mathbb{R}^n} v^{2^*}\, dx = \left(\frac{1}{K(n,0)}\right)^{\frac{2^*}{2^*-2}}$$

And hence we have (4.110). This ends Step 5. □

Proceeding as in Step 3, for every $1 \leq j \leq n$ we have as $\epsilon \to 0$

$$\int_{\mathcal{T}\left(B_{\mathcal{T}^{-1}(x_\epsilon)}(r_\epsilon/2)\right)} \partial_j a(x)\ u_\epsilon^2(x)\ dx = \begin{cases} O(\mu_\epsilon) & \text{for } n = 3, \\ O\left(\mu_\epsilon^2 \log\left(\frac{r_\epsilon}{k_\epsilon}\right)\right) & \text{for } \ n = 4, \\ O\left(\mu_\epsilon^2\right) & \text{for } \ n \geq 5. \end{cases} \tag{4.111}$$

Using the Pohozaev identity (4.105) and the preceding estimates obtained after Steps 1 to 3, noting that $r_\epsilon = d(x_\epsilon, \partial\Omega) = (1 + o(1))|x_{\epsilon,1}|$, we then obtain that

$$d(x_\epsilon, \partial\Omega) = (1 + o(1))|x_\epsilon| \text{ as } \epsilon \to 0 \text{ when } n = 3 \text{ or } a \equiv 0.$$

When $n = 4$, Then as $\epsilon \to 0$

$$\frac{s_\epsilon}{4} \frac{(x_\epsilon)_1}{|x_\epsilon|^2} \left(K(4,0)^{-2} + o(1)\right) + \frac{\mu_\epsilon^2}{r_\epsilon^3} \left(32\omega_3 + o(1)\right) = O\left(\mu_\epsilon^2 \log\left(\frac{r_\epsilon}{\mu_\epsilon}\right)\right)$$

Finally, when $n \geq 5$, we get as $\epsilon \to 0$

$$\frac{s_\epsilon(n-2)}{2n} \frac{(x_\epsilon)_1}{|x_\epsilon|^2} \left(K(n,0)^{-n/2} + o(1)\right) + r_\epsilon^{-1} \left(\frac{\mu_\epsilon}{r_\epsilon}\right)^{n-2} \left(\frac{n^{n-2}(n-2)^n \omega_{n-1}}{2^{n-1}} + o(1)\right) = O\left(\mu_\epsilon^2\right)$$

Plugging together these estimates and the estimates after Steps 1 to 3 yields Theorem 4.10. □

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